# Edge Coloring with Minimum Reload/Changeover Costs*

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In an edge colored graph, traversal cost occurs at a vertex along a path when consecutive edges with different colors are traversed. We focus in this paper on two problems which aim to find a proper edge-coloring of a graph (together with some other constraints) so that the total traversal cost is minimized. We present hardness results as well as polynomial-time solvable special cases.

**Keywords:** Changeover cost, reload cost, approximation algorithms, edge coloring.

## 1 Introduction

The cost incurred while traversing a vertex via two consecutive edges of different colors is called *traversal cost*. This cost appeared in the literature under the names of *reload cost* and *changeover cost*. The value of the traversal cost depends on the colors of the incident traversed edges. The concept of traversal cost first appeared in [9] under the name of reload cost. Although this concept has numerous applications, it has hitherto received little attention. Some network optimization problems related to this concept have been studied in the literature. Some examples are the minimum reload cost diameter problem [9, 1], the minimum reload cost cycle cover problem [3], the minimum changeover cost arborescence problem [2, 6], as well as numerous path, tour and flow problems [4]. All of these problems focus on edge-colored graphs, where the coloring is given as input to the problem.

In this paper, we take a different approach and focus on proper edge coloring of a given graph such that the total traversal cost is minimized. To the best of our knowledge, this paper is the first study about edge coloring under the the traversal cost concept. The problems we study have important applications in telecommunications. Recently, cognitive radio networks (CRN) have gained increasing attention in the communication networks research community. Unlike other wireless technologies, CRNs are envisioned to operate in a wide range of frequencies. Therefore, switching from one frequency band to another frequency band in a CRN has a significant cost in terms of delay and power consumption [5]. The problems we study in this paper have important applications in cognitive radio networks such as optimally allocating frequencies to wireless links so that the total energy consumption or switching delay is minimized.

## 2 Preliminaries

Given an undirected graph $G = (V(G), E(G))$, we consider edge colorings $\chi : E(G) \to X$ of $G$ where the colors are taken from a set $X$ and edges incident to the same vertex are assigned different colors.

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Without loss of generality we assume $X = \{1, 2, \ldots, |X|\}$. Since, by Vizing theorem, every graph is $\Delta(G) + 1$ edge colorable, we also assume that $|X| \geq \Delta(G) + 1$ so that $G$ is colorable with colors from $X$.

We follow the notation of [9] where the concept of reload cost was defined, however with a different naming. We use the term *traversal cost* instead of *reload cost* for reasons that will be apparent later. The traversal costs are given by a nonnegative function $tc : X^2 \rightarrow \mathbb{R}^+ \cup \{0\}$ satisfying
i) $tc(i, j) = tc(j, i)$ for every $i, j \in X$, and
ii) $tc(i, i) = 0$ for every $i \in X$.

Let $P = (e_1, e_2, \ldots, e_\ell)$ a path of length $\ell$ of $G$. We denote by $tr(P) = \{\{e_i, e_{i+1}\} : 1 \leq i < \ell\}$ the set of traversals of $P$. The traversal cost associated with a traversal $t_i = \{e_i, e_{i+1}\}$ of $P$ is $tc(t_i) \stackrel{def}{=} tc(\chi(e_i), \chi(e_{i+1}))$. The traversal cost associated with $P$ is $tc(P) \stackrel{def}{=} \sum_{t \in tr(P)} tc(t)$. Note that the traversal cost of $P$ is zero whenever its length is zero or one. Therefore, we assume that all the paths under consideration have length at least 2.

Let $\mathcal{P}$ be the set of paths. The set of traversals of $\mathcal{P}$ is $tr(\mathcal{P}) \stackrel{def}{=} \bigcup_{P \in \mathcal{P}} tr(P)$. The *reload cost* of a set of paths $\mathcal{P}$ is

$$rc(\mathcal{P}) \stackrel{def}{=} \sum_{P \in \mathcal{P}} tc(P) = \sum_{P \in \mathcal{P}} \sum_{t \in tr(P)} tc(t),$$

and its *changeover cost* is

$$cc(\mathcal{P}) \stackrel{def}{=} \sum_{t \in tr(\mathcal{P})} tc(t).$$

We assume without loss of generality that $E(G) = \cup_{P \in \mathcal{P}} E(P)$, i.e. every edge of $G$ is used by at least one path. We note that whenever every traversal is in at most one path of $\mathcal{P}$, we have $rc(\mathcal{P}) = cc(\mathcal{P})$. In particular, this holds when $\mathcal{P}$ is a set (as opposed to a multi-set) of paths with length 2.

The minimum reload cost edge coloring (MinRCEC) and minimum changeover cost edge coloring (MinCCEC) problems aim to find an edge coloring of $G$ with minimum reload and changeover costs, respectively. Formally,

MinRCEC $(G, \mathcal{P}, X, tc)$
**Input:** A graph $G$, a set of paths $\mathcal{P}$ of $G$, a set $X$ of at least $\Delta(G) + 1$ colors, a traversal cost function $tc : X^2 \rightarrow \mathbb{R}^+ \cup \{0\}$
**Output:** A proper edge coloring $\chi : E(G) \rightarrow X$
**Objective:** Minimize $rc(\mathcal{P})$.

MinCCEC $(G, \mathcal{P}, X, tc)$
**Input:** A graph $G$, a set of paths $\mathcal{P}$ of $G$, a set $X$ of at least $\Delta(G) + 1$ colors, a traversal cost function $tc : X^2 \rightarrow \mathbb{R}^+ \cup \{0\}$
**Output:** A proper edge coloring $\chi : E(G) \rightarrow X$
**Objective:** Minimize $cc(\mathcal{P})$.

Given a tree $T$ and a vertex $r \in V(T)$, let $\mathcal{P}(T, r) \stackrel{def}{=} \{p_T(r, v) : v \in V(T) \setminus \{r\}\}$ where $p_T(r, v)$ is the unique path between $r$ and $v$ in $T$. The reload and changeover costs of $T$ rooted at $r$ are $rc(T, r) \stackrel{def}{=} rc(\mathcal{P}(T, r))$ and $cc(T, r) \stackrel{def}{=} cc(\mathcal{P}(T, r))$, respectively. Given a graph $G$ and a vertex $r$ in the graph, the minimum reload cost path tree edge coloring (MinRCPTEC) and minimum changeover cost arborescence edge coloring (MinCCAEC) problems aim to find a spanning tree $T$ rooted at $r$ and a proper edge coloring of $T$ with minimum total reload and changeover cost, respectively. Formally,

MinRCPTEC $(G, r, X, tc)$
**Input:** A graph $G$, a set of paths $\mathcal{P}$ of $G$, a vertex $r$ of $G$, a set $X$ of at least $\Delta(G) + 1$ colors, a traversal cost function $tc : X^2 \rightarrow \mathbb{R}^+ \cup \{0\}$
**Output:** A spanning tree $T$ of $G$ and a proper edge coloring $\chi : E(T) \mapsto X$
**Objective:** Minimize $rc(T, r)$.

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| MinCCAEC $(G, r, X, tc)$<br>**Input:** A graph $G$, a set of paths $\mathcal{P}$ of $G$, a vertex $r$ of $G$, a set $X$ of at least $\Delta(G)+1$ colors, a traversal cost function $tc : X^2 \rightarrow \mathbb{R}^+ \cup \{0\}$<br>**Output:** A spanning tree $T$ of $G$ and a proper edge coloring $\chi : E(T) \mapsto X$<br>**Objective:** Minimize $cc(T, r)$. |

**Approximation Algorithms.** Let $\Pi$ be a minimization problem and $\rho \geq 1$. A (feasible) solution $s$ of an instance $I$ of $\Pi$ is a *$\rho$-approximation* if its objective function value $O_\Pi(s)$ is at most a factor $\rho$ of the optimal objective function value $O^*_\Pi(I)$ of $I$, i.e., $O_\Pi(s) \leq \rho \times O^*_\Pi(I)$. An algorithm $ALG$ is said to be a *$\rho$-approximation algorithm* for a minimization problem $\Pi$ if $ALG$ returns a $\rho$-approximation for every instance $I$ of $\Pi$ supplied to it. $\Pi$ is said to be *$\rho$-inapproximable* if there is no polynomial-time $\rho$-approximation algorithm for it unless $P = NP$.

**The Lightest $k$-Subgraph Problem:**
Given an edge weighted graph $G$, the k-Lightest Subgraph problem is to find an induced subgraph of $G$ on $k$ vertices, with minimum total edge weight. This problem is NP-Hard in the strong sense even when the graph is a complete graph and the edge weights are either 1 or 2 [8].

## 3 Hardness Results

**Theorem 1.** MinCCEC *and* MinRCEC *are inapproximable within any polynomial-time computable function $f(|\mathcal{P}|)$.*

*Proof.* The proof is by reduction from the minimum chromatic index problem. The chromatic index of a graph $G$ is either $\Delta(G)$ or $\Delta(G)+1$. However, it in NP-Complete to decide between these two values [7]. Given a graph $G$ we construct an instance $(G, \mathcal{P}, X, tc)$ where $\mathcal{P}$ consists of all the distinct paths of length 2 of $G$, $|X| = \Delta(G)+1$, and

$$tc(i,j) = \begin{cases} 0 & \text{if } i = j \\ M & \text{if } i \neq j \text{ and } \max(i,j) = \Delta(G)+1 \\ 1 & \text{otherwise} \end{cases}$$

where $M = |\mathcal{P}| \cdot f(|\mathcal{P}|)$. If $G$ is $\Delta(G)$ edge-colorable, i.e. the minimum chromatic index of $G$ is $\Delta(G)$, for an edge coloring $\chi$ with $\Delta(G)$ colors we get $rc(\mathcal{P}) = cc(\mathcal{P}) = |\mathcal{P}|$. Therefore, an $f(|\mathcal{P}|)$-approximation algorithm $\mathcal{A}$ for either problem will return at most $|\mathcal{P}| \cdot f(|\mathcal{P}|)$. On the other hand, for any edge coloring with $\Delta(G)+1$ colors we have $rc(\mathcal{P}) = cc(\mathcal{P}) \geq |\mathcal{P}| + M - 1 = |\mathcal{P}| + |\mathcal{P}| \cdot f(|\mathcal{P}|) - 1 > |\mathcal{P}| \cdot f(|\mathcal{P}|)$. Therefore, $G$ is $\Delta(G)$ edge-colorable if and only if $\mathcal{A}$ returns at most $|\mathcal{P}| \cdot f(|\mathcal{P}|)$. □

We now show that both problems are NP-Complete even in very simple graphs that are in particular $\Delta(G)$ edge-colorable, namely stars.

**Theorem 2.** MinCCEC *and* MinRCEC *are* NP-*Hard in the strong sense even when $tc(i,j) \in \{1,2\}$ for every distinct pair $i, j$ and $G$ is a star.*

*Proof.* We show a reduction from the Lightest $k$-Subgraph problem which is NP-Hard in the strong sense even on complete graphs with edge weights either 1 or 2. Given such an instance $(K, w)$ of Lightest $k$-Subgraph where $K$ is a clique and $w$ is the edge weight function, we build the following instance: $G$ is a star on $k+1$ vertices. $\mathcal{P}$ consists of the $\binom{k}{2}$ paths between every pair of leaves, $|X| = |K|$, and $tc(i,j) = w(i,j)$. The cost of a solution of MinCCEC (or MinCCEC) on this instance is equal to the weight of a clique on $k$ vertices of $K$. □

**Theorem 3.** MinCCAEC *and* MinRCPTEC *are* NP-*Hard in the strong sense.*

*Proof.* We prove this theorem by a reduction from the set cover problem. □

## 4 Polynomial-time Solvable Cases

**Theorem 4.** MinCCEC *problem is solvable in polynomial-time when $G$ is a tree, and a particular vertex $r$ of $G$ is an endpoint of every path $P \in \mathcal{P}$.*

*Proof.* We proposing a dynamic programming algorithm that traverses the tree and finds a minimum weight perfect matching of an auxiliary graph at each vertex. □

We note that in the MinCCAEC problem when $G$ is a tree, there is only one spanning tree, and we get is a special case of the MinCCEC problem where $G$ is a tree and a particular vertex $r$ of $G$ is the source vertex of all paths. Therefore,

**Corollary 5.** MinCCAEC *problem is solvable in polynomial-time for trees.*

**Theorem 6.** MinCCEC *problem is solvable in polynomial-time when $G$ is a tree and $|X|^{\Delta(G)}$ is a polynomial, i.e., when $\Delta(G)$ is poly-logarithmic.*

*Proof.* We prove this theorem by proposing a dynamic programming algorithm and using the fact that there are at most $\binom{|X|}{2}$ color traversals at each vertex in this special case. □

**Theorem 7.** MinCCAEC *problem is solvable in polynomial-time for cactus graphs.*

*Proof.* We prove this theorem by proposing a dynamic programming-based algorithm. □

**Theorem 8.** MinCCAEC *problem is solvable in polynomial-time for graphs $G$ with $|E(G)| - |V(G)|$ is bounded by some constant.*

*Proof.* This is done by trying all the possible spanning trees which are at most $|E(G)|^{|E(G)|-|V(G)|}$ in number. □

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