Equimatchable Graphs are C_{2k+1}-free for $k \geq 4$*

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A graph is equimatchable if all of its maximal matchings have the same size. Equimatchable graphs are extensively studied in the literature mainly from structural point of view. Here we provide, to the best of our knowledge, the first family of forbidden subgraphs of equimatchable graphs. Since equimatchable graphs are not hereditary, the task of finding forbidden subgraphs requires the use of structural results from previous works.

1 Introduction

A graph G is equimatchable if every maximal matching of G has the same size. Equimatchable graphs were first considered independently in [3] and [5] in 1974. However, they were formally introduced in 1984 [2]. In this work we provide, to the best of our knowledge, the first family of forbidden induced subgraphs of equimatchable graphs. Namely, we show that equimatchable graphs do not contain odd cycles of length at least nine. Our proof is based on the Gallai-Edmonds decomposition of equimatchable graphs given in [2] and the structure of factor-critical equimatchable graphs [1].

Let us first point out that equimatchable graphs do not admit a forbidden subgraph characterization since being equimatchable is not a hereditary property, that is, it is not necessarily preserved by induced subgraphs. In light of this information, finding forbidden subgraphs for equimatchability boils down to finding graphs which are not only non-equimatchable, but are also not an induced subgraph of an equimatchable graph. This task is indeed more complicated than finding "minimally non-equimatchable" graphs and thus requires different methods.

2 Preliminaries

Given a simple graph (no loops or parallel edges) $G = (V(G), E(G))$ and a vertex v of G, we denote by $N(v)$ the set of neighbors (or the *open neighborhood*) of v in G. The *closed*

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neighborhood of v is $N[v] = N(v) \cup \{v\}$. For a graph G and $U \subseteq V(G)$, we denote by $G[U]$ the subgraph of G induced by U. The union $G \cup G'$ of two graphs G, G' is $(V(G) \cup V(G'), E(G) \cup E(G'))$ and their difference is $G \setminus G' = G[V(G) \setminus V(G')]$. We denote by $|G| = |V(G)|$ the order of the graph G and by $d_G(u, v)$ the distance between the vertices u and v in G. For a set X and a singleton $Y = \{y\}$, we denote $X \cup Y$ and $X \setminus Y$ by $X + y$ and $X - y$, respectively. We denote by P_n, C_n and K_n the path, the cycle and the complete graph on n vertices, respectively, and by $K_{n,m}$ the complete bipartite graph with bipartition of sizes n and m.

A *matching* of a graph G, is a subset $M \subseteq E(G)$ of pairwise non-adjacent edges. We denote by $V(M)$ the set of endpoints of M. A vertex v of G is *saturated* by M if $v \in V(M)$ and *exposed* by M otherwise. A matching M is *maximal* in G if no other matching of G contains M. A matching is a *maximum* matching of G if it is a matching of maximum cardinality. A matching is a *perfect* matching of G if $V(M) = V(G)$.

A graph G is *equimatchable* if every maximal matching of G has the same cardinality. A graph G is *randomly matchable* if every matching of G can be extended to a perfect matching. In other words, randomly matchable graphs are equimatchable graphs admitting a perfect matching. A graph G is *factor-critical* if $G - u$ has a perfect matching for every vertex u of G.

Now, we proceed with the Gallai-Edmonds decomposition theorem, which gives an important characterization of a graph based on its maximum matchings.

Theorem 1. *(Gallai-Edmonds decomposition) [4] Let G be a graph, $D(G)$ the set of vertices of G that are not saturated by at least one maximum matching, $A(G)$ the set of vertices of $V(G) \setminus D(G)$ with at least one neighbor in $D(G)$, and $C(G) = V(G) \setminus (D(G) \cup A(G))$. Then, the connected components of $G[D(G)]$ are factor-critical, $G[C(G)]$ has a perfect matching, and every maximum matching of G matches every vertex of $A(G)$ to a vertex of a distinct component of $G[D(G)]$.*

We now state a few results from the literature that will be useful in our proofs.

Lemma 2. *[6] A connected graph is randomly matchable if and only if it is isomorphic to K_{2n} or $K_{n,n}$ $(n \geq 1)$.*

Lemma 3. *[2] Let G be a connected equimatchable graph with no perfect matching. Then $C(G) = \emptyset$ and $A(G)$ is an independent set of G.*

Theorem 4. *(Theorem 3 in [2]) Let G be a connected, equimatchable, and non factor-critical graph without a perfect matching. Let D_i be a connected component of $G[D(G)]$ with at least three vertices. Then, either D_i has exactly one neighbor in $A(G)$ and D_i is P_4-free or D_i contains a cut vertex of G separating D_i into connected components $D_{i,j}$, each of which is P_4-free.*

Theorem 3 of [2] provides the exact structure of the components D_i and $D_{i,j}$, which we omit here for brevity. The fact that every component mentioned therein is P_4-free can be easily derived from the rather involved statement of the theorem.

A matching M *isolates* v in G if v is an isolated vertex of $G \setminus V(M)$. We use the following lemma in our proofs.

Lemma 5. *[1] Let G be a connected, factor-critical, equimatchable graph and M be a matching isolating v. Then $G \setminus (V(M) + v)$ is randomly matchable.*

3 Forbidden Subgraphs of Equimatchable Graphs

Using Theorem 4, we first show that if an equimatchable graph contains an odd cycle of length at least five, then it is factor-critical.

Lemma 6. *Let G be an equimatchable and non factor-critical graph with an induced subgraph C isomorphic to a cycle C_m for some $m \geq 5$. Let also D_i be a factor-critical component in the Gallai-Edmonds decomposition of G. Then $|V(C) \cap V(D_i)| \leq 1$.*

Proof. We show that this contradicts Theorem 4. □

Lemma 7. *If G is an equimatchable graph with an induced subgraph C isomorphic to a cycle C_{2k+1} for some $k \geq 2$, then G is factor-critical.*

Proof. By using lemmata 3 and 7 we find a bipartite graph containing C. A contradiction. □

The following observation gives us some insight about the structure of the intersection of a randomly matchable graph with a path.

Observation 8. *Let P be a path of a graph G and H an induced subgraph of G isomorphic to a K_{2n} or a $K_{n,n}$. Then, if $H[V(P)]$ is not connected then H is a $K_{n,n}$ and $H[V(P)]$ is an independent set, otherwise $H[V(P)]$ has at most 3 vertices.*

Given a factor-critical equimatchable graph, we will also need the following construction of a special isolating matching with some additional properties.

Lemma 9. *Let v be a vertex of an equimatchable and factor-critical graph G, and let $C \subseteq V(G)$. There is a set of three vertex disjoint matchings M_1, M_2, M_3 and a partition of $N(v)$ into N_1, N_2, N_3 such that:*
i) $M_1 \cup M_2 \cup M_3$ isolates v,
ii) M_1 is a perfect matching of N_1,
iii)M_2 matches N_2 to some N_2' such that $N_2' \cap C = \emptyset$,
iv) M_3 matches N_3 to some $N_3' \subseteq C \setminus N[v]$,
v) $N_2 \cup N_3$ is an independent set,
vi) $N(N_3) \subseteq N_1 \cup N_2' \cup C + v$.

Proof. Since G is factor-critical, there is a matching isolating v. We provide an algorithm which constructs the sets N_1, N_2, N_3 and the matchings M_1, M_2, M_3 starting from such a matching, greedily augmenting N_1, and then greedily augmenting M_2. Figure 1 depicts the sets N_1, N_2, N_3 and the matching constructed by this algorithm. □

It is easy to verify that C_{2k+1} is equimatchable if and only if $k \leq 3$. In the sequel we prove a stronger result. Namely, C_{2k+1} is an induced subgraph of an equimatchable graph if and only if $k \leq 3$.

Theorem 10. *Equimatchable graphs are C_{2k+1}-free for any $k \geq 4$.*

Proof. Let G be an equimatchable graph and let C be an induced odd cycle of G with at least 9 vertices. Then, by Lemma 7, G is factor-critical. Therefore, every maximal matching of G leaves exactly one vertex exposed. Therefore, a matching M such that $G \setminus V(M)$ has at least 2 odd components, constitutes a contradiction. Our proof is based on building such matchings.

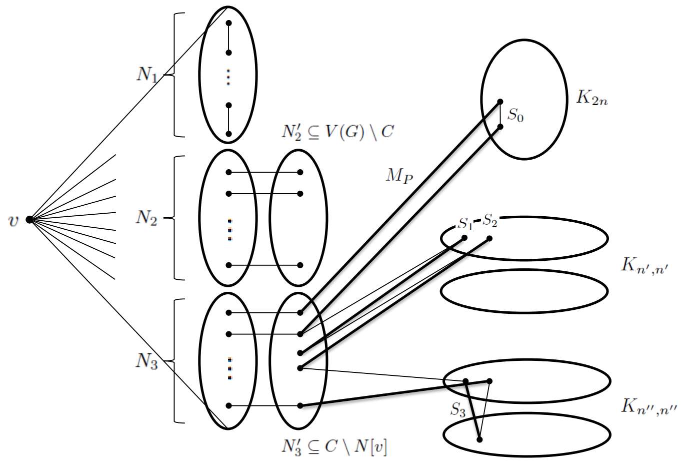

Figure 1: The matching $M_1 \cup M_2 \cup M_3$ isolating v, and the unique perfect matching M_P of P.

Let v be any vertex of C, and let $M_1, M_2, M_3, N_1, N_2, N_3$ be the matchings and the sets of vertices whose existence are guaranteed by Lemma 9. Let $M \stackrel{def}{=} M_1 \cup M_2 \cup M_3$, and let $P = C \setminus N[v]$ denote the path isomorphic to a P_{2k-2} obtained by the removal of v and its two neighbors from the cycle C. Let M_P be the unique perfect matching of P (see Figure 1).

Using Observation 8, a simple counting shows that $|C| \leq 4|N_3| + 6$. Then, $N_3 \neq \emptyset$ for the cycles under consideration. If $|C| \geq 15$ then $|N_3| \geq 3$. Consider a matching $M' = M_1 \cup M_2 \cup M_P + \{v, u\}$ where u is any vertex of N_3. Every vertex of $N_3 - u$ is isolated in $G \setminus V(M')$ by Lemma 9 item vi). Since $|N_3 - u| \geq 2$, M' constitutes a contradiction as described above. In the full version, we build specific matchings for the case $|N_3| \in \{1, 2\}$ to conclude the proof for the cycles C_9, C_{11} and C_{13}. □

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