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Equimatchable graphs are C_{2k+1}-free for $k \geq 4$[☆]

Cemil Dibek[a], Tınaz Ekim[a], Didem Gözüpek[c], Mordechai Shalom[a,b,*]

[a] Department of Industrial Engineering, Bogazici University, Istanbul, Turkey
[b] TelHai Academic College, Upper Galilee, 12210, Israel
[c] Department of Computer Engineering, Gebze Technical University, Kocaeli, Turkey

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ABSTRACT

A graph is equimatchable if all of its maximal matchings have the same size. Equimatchable graphs are extensively studied in the literature mainly from structural point of view. Here we provide the first family of forbidden subgraphs of equimatchable graphs. Since equimatchable graphs are by definition not hereditary, this task of finding forbidden subgraphs requires the use of structural results from previous works.

1. Introduction

A graph G is equimatchable if every maximal matching of G has the same size. Equimatchable graphs were first considered independently in [7,11,13] in 1974. However, they were formally introduced in 1984 [10] where the authors provide a structural characterization of equimatchable graphs yielding a polynomial-time recognition algorithm. A more efficient recognition algorithm is then given in [1]. It also follows from [10] that any 2-connected equimatchable graph which is not randomly matchable is either bipartite, or factor-critical, and these two cases are disjoint. Other studies on equimatchable graphs focus on the structure of these graphs. Factor-critical equimatchable graphs with vertex connectivity 1 and 2 are characterized in [5] where it is also shown that every 2-connected factor-critical equimatchable graph is Hamiltonian. In [9], all equimatchable graphs that are 3-connected planar or 3-connected cubic are determined. In [8], it is shown that equimatchable graphs with fixed genus have bounded size, and in [6], equimatchable graphs with girth at least 5 are characterized. More recently, 2-connected equimatchable graphs embeddable on various surfaces are studied in [4], and a description of k-connected equimatchable factor-critical graphs is given in [3]. A graph G is *well-covered* if every maximal independent set of G has the same size. Clearly, a graph is equimatchable if and only if its line graph is well-covered. This close relationship between equimatchable graphs and well-covered graphs allows us in particular to test in polynomial time whether a line graph, or more generally, a claw-free graph, is well-covered [14].

In this work we provide, to the best of our knowledge, the first family of forbidden induced subgraphs of equimatchable graphs. Namely, we show that equimatchable graphs do not contain odd holes of length at least nine. Our proof is based on the Gallai–Edmonds decomposition of equimatchable graphs given in [10] and the structure of factor-critical equimatchable graphs [4].

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* Corresponding author at: TelHai Academic College, Upper Galilee, 12210, Israel.
E-mail addresses: cemil.dibek@boun.edu.tr (C. Dibek), tinaz.ekim@boun.edu.tr (T. Ekim), didem.gozupek@gtu.edu.tr (D. Gözüpek), cmshalom@telhai.ac.il (M. Shalom).

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Let us first point out that equimatchable graphs do not admit a forbidden subgraph characterization since being equimatchable is not a hereditary property, that is, it is not necessarily preserved by induced subgraphs. For instance, the graph consisting of a cycle on four vertices and a path on three vertices with one endpoint of the path forming a triangle with two vertices of the cycle is equimatchable; however, it contains a triangle with one pending vertex, which is not equimatchable. In light of this information, finding forbidden subgraphs for equimatchability boils down to finding graphs which are not only non-equimatchable, but are also not an induced subgraph of an equimatchable graph. This task is indeed more complicated than finding "minimally non-equimatchable" graphs and thus requires different methods.

Using the above observation, it follows from the characterization of equimatchable graphs with girth at least five given in [6] that no connected bipartite graph with girth at least five is forbidden in an equimatchable graph because they can all be extended to an equimatchable graph by adding a leaf to each one of the vertices in one side of the bipartition. This result implies in particular that even cycles are not forbidden (noting that a cycle on four vertices is equimatchable).

2. Preliminaries

2.1. Notation and definitions

We use standard terminology and notation for graph theory, see for instance [2]. We denote by $N(v)$ and $N[v]$ the open and closed neighborhood of v, respectively on some given graph. A subset of at least 4 vertices inducing a cycle is termed a *hole*. For a set X and a singleton $Y = \{y\}$, we denote $X \cup Y$ and $X \setminus Y$ by $X + y$ and $X - y$, respectively. We denote by P_n, C_n and K_n the path, the cycle and the complete graph, respectively, on n vertices, and by $K_{n,m}$ the complete bipartite graph with bipartition of sizes n and m. For two graphs G and H, G is *H-free* if it does not contain H as an induced subgraph.

A *matching* of a graph G is a subset $M \subseteq E(G)$ of pairwise non-adjacent edges. We denote by $V(M)$ the set of endpoints of M. A vertex v of G is *saturated* by M if $v \in V(M)$ and *exposed* by M otherwise. A matching M is *maximal* in G if no other matching of G contains M. A matching is a *maximum* matching of G if it is a matching of maximum cardinality. A matching M is a *perfect* matching of G if $V(M) = V(G)$. A graph G is *factor-critical* (or *hypomatchable*) if $G - u$ has a perfect matching for every vertex u of G.

A graph G is *equimatchable* if every maximal matching of G has the same cardinality. A graph G is *randomly matchable* if every matching of G can be extended to a perfect matching. In other words, randomly matchable graphs are equimatchable graphs admitting a perfect matching.

2.2. Hereditary equimatchable graphs

In this section we continue the discussion on the non-hereditary property of being equimatchable, mentioned in Section 1. As a result of that discussion, to obtain a characterization by forbidden subgraphs, one should require equimatchability not only for the graph itself but also for all of its induced subgraphs, thus introducing the notion of *hereditary equimatchable* graphs.

A *diamond* is a C_4 with an added chord, whereas a *paw* is a triangle (C_3) with one pendant vertex. The following shows that hereditary equimatchable graphs form a rather small subclass of equimatchable graphs (only very slightly (and naturally) generalizing randomly matchable graphs characterized in Lemma 3):

Proposition 1. *The following are equivalent:*

(i) *G is a connected hereditary equimatchable graph.*
(ii) *G is (P_4, diamond, paw)-free.*
(iii) *G is a complete graph or a complete bipartite graph.*

Proof. It is easy to check that none of the graphs P_4, diamond and paw is equimatchable; therefore, i $\Rightarrow$ ii. Moreover, every complete graph and every complete bipartite graph is equimatchable, thus iii $\Rightarrow$ i.

ii $\Rightarrow$ iii: Let G be a (P_4, diamond, paw)-free connected graph on n vertices. Let us show by induction on n that G is a complete graph or a complete bipartite graph. The claim holds for $n \leq 4$ since the only connected (P_4, diamond, paw)-free graphs on at most 4 vertices are K_1, K_2, K_3, $K_{1,2}$, $K_{1,3}$, C_4 and K_4.

Let $n \geq 5$, and v be a vertex of G. If $G - v$ is disconnected, then G is a $K_{1,n-1}$ since as soon as $G - v$ contains a component with at least two vertices, v is in a P_4 or a paw. Otherwise, by the induction hypothesis $G' = G - v$ is either a complete graph or a complete bipartite graph.

If G' is a complete graph and G is not a complete graph, then taking v, a neighbor of v, a non-neighbor of v, and a fourth vertex induces either a paw or a diamond.

If G' is a complete bipartite graph, consider a neighbor u of v. Then, v is adjacent to every vertex w in the same part as u, since otherwise v, u, w and any vertex from the other part of G' form either a paw or a P_4. Besides, we observe that v is non-adjacent to any vertex u' in the other part of G', since otherwise v, u, u' and a fourth vertex adjacent to one of u, u' form either a paw or a diamond. Therefore, G is a complete bipartite graph. $\square$

2.3. Related structural results

We start with the Gallai–Edmonds decomposition theorem, which gives an important characterization of a graph based on its maximum matchings.

Theorem 2 (*Gallai–Edmonds Decomposition [12]*). *Let G be a graph, $D(G)$ the set of vertices of G that are not saturated by at least one maximum matching, $A(G)$ the set of vertices of $V(G) \setminus D(G)$ with at least one neighbor in $D(G)$, and $C(G) \stackrel{def}{=} V(G) \setminus (D(G) \cup A(G))$. Then:*

(i) *the components of $G[D(G)]$ are factor-critical,*
(ii) *$G[C(G)]$ has a perfect matching,*
(iii) *every maximum matching of G matches every vertex of $A(G)$ to a vertex of a distinct component of $G[D(G)]$.*

We now state a few results from the literature that will be useful in our proofs.

Lemma 3 (*[15]*). *A connected graph is randomly matchable if and only if it is isomorphic to a K_{2n} or a $K_{n,n}$ $(n \geq 1)$.*

Lemma 4 (*[10]*). *Let G be a connected equimatchable graph with no perfect matching. Then $C(G) = \emptyset$ and $A(G)$ is an independent set of G.*

Theorem 5 (*Theorem 3 in [10]*). *Let G be a connected, equimatchable, and non factor-critical graph without a perfect matching. Let D_i be a component of $G[D(G)]$ with at least three vertices. Then one of the following holds:*

(i) *D_i has exactly one neighbor in $A(G)$ and D_i is P_4-free.*
(ii) *D_i contains a cut vertex of G separating D_i into components $D_{i,j}$, each of which is P_4-free.*

Theorem 3 of [10] provides the exact structure of the components D_i and $D_{i,j}$, which we omit here for brevity. A precise consideration of these components reveals that these components are P_4-free.

A matching M *isolates* v in G if v is an isolated vertex of $G \setminus V(M)$. A *minimal* isolating matching for v is a matching that isolates v, but no proper subset of it does so. We use the following lemma in our proofs.

Lemma 6 (*[4]*). *Let G be a connected, factor-critical, equimatchable graph and M be a minimal isolating matching for v. Then $G \setminus (V(M) + v)$ is randomly matchable.*

3. Forbidden subgraphs of equimatchable graphs

Using Theorem 5, we first show that if an equimatchable graph contains an odd hole, then it is factor-critical.

Lemma 7. *Let G be an equimatchable graph that does not admit a perfect matching and is not factor-critical. Let D_i be a factor-critical component in the Gallai–Edmonds decomposition of G, and let C be a hole of G with at least 5 vertices. Then $|V(C) \cap V(D_i)| \leq 1$.*

Proof. Since C is 2-connected, $C - v$ is in a component of $G - v$. Clearly, $C - v$ contains an induced P_4. Suppose that some factor-critical component D_i contains two vertices of C. Then, $D_i - v$ contains $C - v$. Thus, $D_i - v$ contains an induced P_4. Noting that both cases of Theorem 5 imply that $D_i - v$ is P_4-free, we get a contradiction. □

Lemma 8. *If G is an equimatchable graph with an odd hole, then G is factor-critical.*

Proof. We first note that G does not admit a perfect matching since otherwise it is randomly matchable, and by Lemma 3 every component of G is either a complete graph of even order or a complete bipartite graph both of which are (odd hole)-free.

Suppose that G is not factor-critical. Then it has a (non-trivial) Gallai–Edmonds decomposition. Construct a graph G' by contracting every component D_i of $D(G)$ to a single vertex. The vertices corresponding to the components D_i constitute an independent set of G'. Since G is equimatchable, by Lemma 4, $A(G)$ is an independent set and $C(G) = \emptyset$. Therefore, G' is bipartite. Let C be an odd hole of G (note that here C and $C(G)$ are different; C is an odd hole of G while $C(G)$ denotes the set of vertices defined for Gallai–Edmonds decomposition of G in Theorem 2). Lemma 7 implies that no pair of vertices of C is contracted when G' is constructed from G. Then C is an odd hole of G', contradicting the fact that G' is bipartite. □

The following observation gives us some insight about the structure of the intersection of a randomly matchable graph with a path.

Observation 9. *Let P be an induced path of a graph G and H be an induced subgraph of G isomorphic to a K_{2n} or a $K_{n,n}$. If $H[V(P)]$ is not connected, then H is a $K_{n,n}$ and $H[V(P)]$ is an independent set; otherwise, $H[V(P)]$ has at most 3 vertices.*

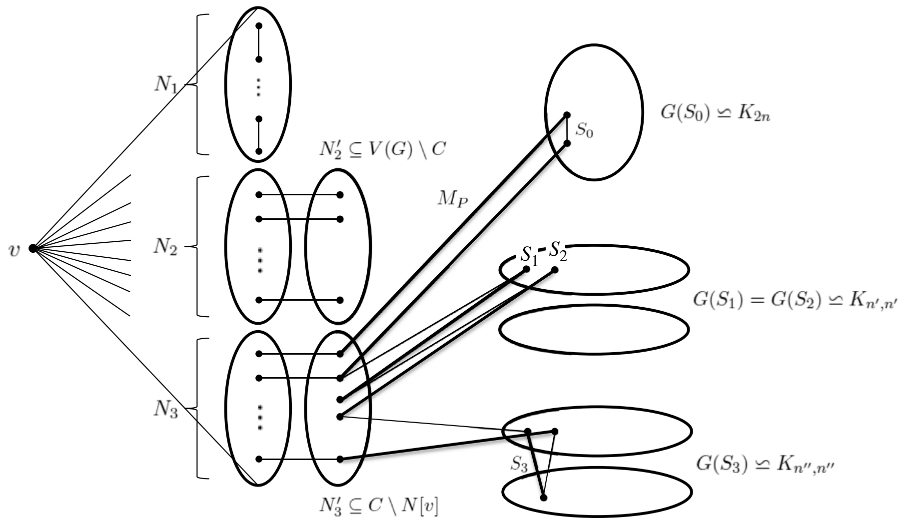

Fig. 1. The matching $M_1 \cup M_2 \cup M_3$ isolating v and the unique perfect matching M_P of P.

Given a factor-critical equimatchable graph, and any subset C of its vertices (not necessarily a hole) we will also need the following construction of a special minimal isolating matching having some properties with respect to C.

Lemma 10. *Let v be a vertex of a factor-critical graph G, and let $C \subseteq V(G)$. There is a set of three vertex disjoint matchings M_1, M_2, M_3 and a near-partition*[1] *$\{N_1, N_2, N_3\}$ of $N(v)$ such that:*

(i) *$M_1 \cup M_2 \cup M_3$ is a minimal isolating matching for v,*
(ii) *M_1 is a perfect matching of N_1,*
(iii) *M_2 matches N_2 to some N'_2 such that $N'_2 \cap C = \emptyset$,*
(iv) *M_3 matches N_3 to some $N'_3 \subseteq C \setminus N[v]$,*
(v) *$N_2 \cup N_3$ is an independent set, and*
(vi) *$N(N_3) \subseteq N_1 \cup N'_2 \cup C + v$.*

Proof. Since G is factor-critical, there is a perfect matching M of $G - v$. Starting from such a matching, we first throw away edges not incident with $N(v)$ to get a minimal isolating matching M of v. We will adapt this matching while maintaining the invariant of M being a minimal isolating matching of v.

As long as there are two adjacent vertices x, y of $N(v)$ such that $xy \notin M$ and both are matched to vertices outside $N(v)$, we replace the two edges of M incident to $\{x, y\}$ by the edge xy. At the end of this procedure, M contains a maximal matching M_1 of $N(v)$. Let N_1 be the set of these endpoints, namely $N_1 = V(M_1)$. Then $N(v) \setminus N_1$ is an independent set.

We further modify $M \setminus M_1$ such that M saturates a minimal subset of C. As long as there is a vertex x matched by $M \setminus M_1$ to a vertex of C and x is adjacent to an unmatched vertex y different from v and not in C, we replace the matching edge incident to x with xy. N_2 (resp. N_3) is the set of vertices of $N(v)$ matched to a vertex not in C (resp. in C).

Algorithm BuildIsolatingMatching greedily constructs the sets N_1, N_2, N_3 and the matchings M_1, M_2, M_3 as described above, and Fig. 1 depicts the sets N_1, N_2, N_3 and these matchings. The proof of the claimed properties follow from the algorithm and its invariants that are stated as comments in Algorithm 1. □

It is easy to verify that C_{2k+1} is equimatchable if and only if $k \leq 3$. In the sequel, we prove a stronger result; namely, C_{2k+1} is not an induced subgraph of an equimatchable graph whenever $k \geq 4$.

Theorem 11. *Equimatchable graphs are C_{2k+1}-free for any $k \geq 4$.*

Proof. Let G be an equimatchable graph and let C be an odd hole of G with at least 9 vertices. Then, by Lemma 8, G is factor-critical. Therefore, every maximal matching of G leaves exactly one vertex exposed. In the rest of the proof, we construct matchings such that the removal of the vertex set of each one of them separates G into at least two odd components, implying the existence of maximal matchings leaving at least two vertices of G exposed, leading to a contradiction.

Let v be any vertex of C, and let $M_1, M_2, M_3, N_1, N_2, N_3$ be the matchings and the sets of vertices whose existence are guaranteed by Lemma 10. Let $M \overset{def}{=} M_1 \cup M_2 \cup M_3$, and let $P = C \setminus N[v]$ denote the path isomorphic to a P_{2k-2} obtained by the

[1] A partition possibly containing empty parts. For instance, if $N(v)$ is an independent set, then the set N_1 would be empty; or if the graph G is K_{2n+1}, then the sets N_2 and N_3 would be empty.

Algorithm 1 BuildIsolatingMatching

Require: G is a factor-critical graph, $v \in V(G)$, $C \subseteq V(G)$
Ensure: $N_1, N_2, N_3, M_1, M_2, M_3$ satisfy the conditions of Lemma 10

$M \leftarrow$ a perfect matching of $G - v$.
while $\exists e \in M$ joining two vertices x, y of $V(G) \setminus N(v)$ **do**
  $M \leftarrow M - e$.
$\triangleright$ M is a minimal isolating matching for v.

$\triangleright$ Use a maximal matching of $N(v)$.
while $\exists xy \in E(G) \setminus M$ such that $x, y \in N(v)$ and x, y matched to vertices not in $N(v)$ **do**
  $e_x \leftarrow$ the edge incident to x in M.
  $e_y \leftarrow$ the edge incident to y in M.
  $M \leftarrow M - e_x - e_y + xy$.
$\triangleright$ M is a minimal isolating matching for v.

$\triangleright$ Partition M into M_1, M_2, M_3 according to the endpoints of the edges.
$M_1 \leftarrow$ all edges $e \in M$ with exactly two endpoints in $N(v)$
$\triangleright$ M_1 is a maximal matching of $N(v)$.
$M_2 \leftarrow$ all edges $e \in M$ with one endpoint in $N(v)$ and the other in $V(G) \setminus C$
$M_3 \leftarrow$ all edges $e \in M$ with one endpoint in $N(v)$ and the other in C
$N_1 \stackrel{def}{=} V(M_1)$, $N_2 \stackrel{def}{=} V(M_2) \cap N(v)$, $N_3 \stackrel{def}{=} V(M_3) \cap N(v)$.
$\triangleright$ $N_2 \cup N_3$ is an independent set.

$\triangleright$ Augment M_2.
while $\exists xy$ such that $x \in N_3$ and $y \notin V(M_1 \cup M_2 \cup M_3) \cup C + v$ **do**
  $e_x \leftarrow$ the edge incident to x in M_3.
  $M_2 \leftarrow M_2 + xy$.
  $M_3 \leftarrow M_3 - e_x$.
$\triangleright$ Every vertex $x \in N_3$ is adjacent only to vertices of $V(M_1 \cup M_2 \cup M_3) \cup C + v$

removal of v and its two neighbors from the cycle C (we note that $C \cap N(v) \subseteq N_1 \cup N_2 \cup N_3$). Recall that $N_3' \subseteq C \setminus N[v] = P$. We denote by M_P the unique perfect matching of P (see Fig. 1).

Let us first show that $|N_3| \leq 2$. Let $u \in N_3$ and consider the matching $M' = M_1 \cup M_2 \cup M_P + uv$. Every vertex of $N_3 - u$ (i.e. at least two vertices) is isolated in $G \setminus V(M')$ by Lemma 10(vi). Therefore, we reach a contradiction of the form described previously.

Since M isolates v, $G \setminus (M + v)$ is randomly matchable. Therefore, every component of $G \setminus (M + v)$ is randomly matchable. By Lemma 3, each such component is either a K_{2n} or a $K_{n,n}$ for some $n \geq 1$. Let $S = P \setminus N_3'$ and let $S_0, S_1, \ldots,$ be the *segments* (i.e. components) of S in the order they are visited when P is traversed in an arbitrary direction. The number of these segments is clearly at most $|N_3'| + 1 = |N_3| + 1$. Each such segment S_i is in a component $G(S_i) \in \mathcal{G}$. By Observation 9, S_i contains at most 3 vertices.

We also note that $N_3 \neq \emptyset$ since otherwise P contains at most one segment with at most 3 vertices, implying that C has at most 6 vertices, a contradiction.

In the rest of the proof we analyze the remaining two cases separately:

- $|N_3| = 2$: Let $N_3 = \{u_1, u_2\}$ and $N_3' = \{w_1, w_2\}$ and $M_3 = \{u_1w_1, u_2w_2\}$. Then, S contains at most three segments S_0, S_1, S_2. If a segment S_i has three vertices, then $G(S_i) \neq G(S_j)$ for any other $j \neq i$, implying that the removal of the vertices of the matching $M_1 \cup M_2 \cup M_P + vu_1$ leaves u_2 isolated and also $G(S_i)$ as an odd component, leading to a contradiction. Therefore, $|S_0|, |S_1|, |S_2| \leq 2$; moreover, S contains at least two segments since otherwise $|S| \leq 2$ implying that $|P| \leq 4$, i.e. $|C| \leq 7$. We consider two cases:
  - One of u_1, u_2 has no neighbors in S: Let without loss of generality u_1 be such a vertex and let $M^{(2)}$ be the matching $M_1 \cup M_2 + vu_2 + w_1x_1 + w_2x_2$ where x_1 and x_2 are chosen from two different components of $\mathcal{G}$. Then $G \setminus V(M^{(2)})$ contains three odd components, namely $\{u_1\}$ and the components containing x_1 and x_2, leading to a contradiction.
  - Each of u_1, u_2 has a neighbor in S: We divide into two cases:
    * One of w_1, w_2 has neighbors in two different components: Let without loss of generality w_1 be such a vertex and consider the matching $M^{(3)} = M_1 \cup M_2 + u_2w_2 + u_1y_1 + w_1x_1$ where y_1 is a neighbor of u_1 in S and x_1 is a neighbor of w_1 in S, not in the same component as y_1. The removal of $V(M^{(3)})$ from G leaves three odd components, namely $\{v\}$ and the two components of $\mathcal{G}$ containing x_1 and y_1.
    * None of w_1, w_2 has neighbors in two different components: In this case there are exactly two components, and w_1, w_2 are adjacent. We have again two cases:

· u_1 and u_2 have neighbors in two distinct components: In this case the removal of the vertices of the matching $M^{(4)} = M_1 \cup M_2 + w_1w_2 + u_1y_1 + u_2y_2$ where y_1 and y_2 are chosen from distinct components, leaves at least three odd components, namely, $\{v\}$ and the components of $\mathcal{G}$.

· The neighbors of u_1 and u_2 in S are in the same component: Assume without loss of generality that these neighbors are in S_1 and w_1, w_2 are adjacent to $G(S_0)$, $G(S_1)$, respectively. In this case, choose a vertex $y_1 \in S_1$ adjacent to u_1, and a vertex $x_2 \in G(S_0)$ adjacent to w_1. Consider the matching $M^{(5)} = M_1 \cup M_2 + u_2w_2 + u_1y_1 + w_1x_2$. $G \setminus V(M^{(5)})$ contains at least three odd components, namely $\{v\}$ and the components of $\mathcal{G}$.

- $|N_3| = 1$: Let $N_3 = \{u\}$ and $N'_3 = \{w\}$. Then, S contains at most two segments S_0, S_1 with $|S_0|, |S_1| \leq 3$, implying that $|P| \leq 7$, i.e. $|C| \leq 10$. Since C is an odd cycle with at least 9 vertices, $|C| = 9$, i.e. $|P| = 6$ and $|S| = 5$. We conclude without loss of generality that $|S_0| = 2$ and $|S_1| = 3$, and $G(S_0) \neq G(S_1)$. Clearly, w is adjacent to both S_0 and S_1. We consider two cases:
  – u has a neighbor in S: Let y be a vertex of S adjacent to u, and let x be a vertex of S adjacent to w and not in the same segment as y. Then, the matching $M_1 \cup M_2 + uy + wx$ is a matching whose removal (together with the edges' endpoints) divides G into at least three odd components. Namely, $\{v\}$ and the components of $\mathcal{G}$.
  – u does not have a neighbor in S: We summarize the properties of u: (a) u is not in C (since otherwise w would have 3 neighbors in C), (b) u is adjacent to $v \in V(C)$ and also to $w \in V(P) \subseteq V(C)$ where w is at distance 4 to v on C, and (c) u is adjacent to at most two vertices of $C - v - w$, namely to the neighbors of v in C (in case they are in N_1). We now rename the vertices of C as $z_0, \ldots, z_8$ such that v is renamed as z_0, its neighbors are z_1 and z_8, the vertices of S_0 are z_2 and z_3, w is z_4, and finally the vertices of S_1 are z_5, z_6 and z_7. By choosing the vertex z_2 as v and repeating the same construction, we either reach a contradiction, thus concluding the proof, or we find a vertex u' adjacent to z_2 satisfying properties (a) through (c). By property (b), u' has a neighbor in C which is at distance 4 to z_2 on C; hence, u' is adjacent to either z_6 or z_7. Therefore, u' is adjacent to both $G(S_0)$ and $G(S_1)$. Since these are two distinct components of $\mathcal{G}$, $u' \notin G(S_0)$ and $u' \notin G(S_1)$. By property (c) u' is possibly adjacent also to $\{z_1, z_3\}$ but not to z_0, i.e. $u' \notin N[v]$. We conclude that $u' \in N'_2$. Let $w' \in N_2$ be the vertex matched to u' by M_2. Now consider the matching $M'' = M_1 \cup M_2 - u'w' + vw' + u'z_2 (= u'v') + z_4z_5 (= wz_5)$. Removing $V(M'')$ from G leaves u isolated, and $G(S_0)$ and $G(S_1)$ as odd components, constituting a contradiction. □

4. Conclusion

Having shown that odd holes of order at least 9 are forbidden for being equimatchable, one can wonder whether there are any other forbidden structures. Unfortunately, we do not have any candidate structure which might possibly be forbidden and have no intuition on whether or not such forbidden structures exist. However, we think that one can apply our approach as a generic method to find an equimatchable graph containing the non-equimatchable structure H under consideration or to prove that such an equimatchable graph does not exist.

A possible future work can be to study the impact of our result in line graphs. Since a graph is equimatchable if and only if its line graph is well-covered, and also a graph contains a hole if and only if its line graph contains a hole, our result implies that well-covered line graphs are C_{2k+1}-free for $k \geq 4$. The question is whether this result can be generalized to well-covered claw-free graphs or any other family containing well-covered line graphs.

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