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Equimatchable claw-free graphs[☆]

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ABSTRACT

A graph is equimatchable if all of its maximal matchings have the same size. A graph is claw-free if it does not have a claw as an induced subgraph. In this paper, we provide the first characterization of claw-free equimatchable graphs by identifying the equimatchable claw-free graph families. This characterization implies an efficient recognition algorithm.

1. Introduction

A graph G is *equimatchable* if every maximal matching of G has the same cardinality. Equimatchable graphs are first considered by Grünbaum [8], Lewin [18], and Meng [19] simultaneously in 1974. They are formally introduced by Lesk et al. in 1984 [16]. Equimatchable graphs can be recognized in polynomial time (see [16] and [2]). From the structural point of view, all 3-connected planar equimatchable graphs and all 3-connected cubic equimatchable graphs are determined by Kawarabayashi et al. [13]. Besides, Kawarabayashi and Plummer showed that equimatchable graphs with fixed genus have bounded size [12], while Frendrup et al. characterized equimatchable graphs with girth at least 5 [7]. Factor-critical equimatchable graphs with vertex connectivity 1 and 2 are characterized by Favaron [5].

A graph G is *well-covered* if every maximal independent set of G has the same cardinality. Well-covered graphs are closely related to equimatchable graphs since the line graph of an equimatchable graph is a well-covered graph. Finbow et al. [6] provide a characterization of well-covered graphs that contain neither 4-cycles nor 5-cycles, whereas Staples [20] provides characterizations of some subclasses of well-covered graphs. A graph is *claw-free* if it does not have a claw as an induced subgraph. Recognition algorithms for claw-free graphs have been presented by Kloks et al. [14], Faenza et al. [4], and Hermelin et al. [11]. Claw-free well-covered graphs have been investigated by Levit and Tankus [17] and by Hartnell and Plummer [10]. However, to the best of our knowledge, there is no previous study in the literature about claw-free equimatchable graphs.

In this paper, we investigate the characterization of claw-free equimatchable graphs. In Section 2, we give some preliminary results. In particular, we show that the case of equimatchable claw-free graphs with even number of vertices reduces to cliques with an even number of vertices or a 4-cycle, and all graphs with odd number of vertices and independence

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number at most 2 are claw-free and equimatchable. We also show that the remaining equimatchable claw-free graphs have (vertex) connectivity at most 3. Based on this fact, in Section 3, we focus on 1-connected, 2-connected (based on a result of Favaron [5]) and 3-connected equimatchable claw-free graphs with odd number of vertices separately. Our full characterization is summarized in Section 4, where we provide a recognition algorithm running in time $\mathcal{O}(m^{1.407})$ where m refers to the number of edges in the input graph.

2. Preliminaries

In this section, after giving some graph theoretical definitions, we mention some known results about matchings in claw-free graphs and develop some tools for our proofs.

Given a simple graph $G = (V(G), E(G))$, a *clique* (resp. *independent set*) of G is a subset of pairwise adjacent (resp. non-adjacent) vertices of G. The *independence number* of G denoted by $\alpha(G)$ is the maximum size of an independent set of G. We denote by $N(v)$ the set of neighbors of $v \in V(G)$. For a subgraph G' of G, $N_{G'}(v)$ denotes $N(v) \cap V(G')$. A vertex v is *complete to* a subgraph G' if $N_{G'}(v) = V(G')$. For $U \subseteq V(G)$, we denote by $G[U]$ the subgraph of G induced by U. For simplicity, according to the context, we will use a set of vertices or the (sub)graph induced by a set of vertices in the same manner. We denote by uv a potential edge between two vertices u and v. Similarly, we denote paths and cycles of a graph as sequences of its vertices. In this work, n denotes the *order* $|V(G)|$ of the graph G. We say that G is an *odd graph* (resp. *even graph*) if n is odd (resp. even). For a set X and a singleton $\{x\}$ we use the shorthand notations $X + x$ and $X - x$ for $X \cup \{x\}$ and $X \setminus \{x\}$, respectively.

We denote by P_p, C_p and K_p the path, cycle, and complete graph, respectively, on p vertices and by $K_{p,q}$ the complete bipartite graph with bipartition sizes p and q. The graph $K_{1,3}$ is termed *claw*. A graph is *claw-free* if it contains no claw as an induced subgraph.

A set of vertices S of a connected graph G such that $G \setminus S$ is not connected is termed a *cut set*. A cut set is *minimal* if none of its proper subsets is a cut set. A *k-cut* is a cut set with k vertices. A graph is *k-connected* if it has more than k vertices and every cut set of it has at least k vertices. The *(vertex) connectivity* of G, denoted by $\kappa(G)$, is the largest number k such that G is k-connected.

A *matching* of a graph G is a subset $M \subseteq E(G)$ of pairwise non-adjacent edges. A vertex v of G is *saturated* by M if $v \in V(M)$ and *exposed* by M otherwise. A matching M is *maximal* in G if no other matching of G contains M. Note that a matching M is maximal if and only if $V(G) \setminus V(M)$ is an independent set. A matching M is a *perfect matching* of G if $V(M) = V(G)$.

A graph G is *equimatchable* if every maximal matching of G has the same cardinality. A graph G is *randomly matchable* if every matching of G can be extended to a perfect matching. In other words, randomly matchable graphs are equimatchable graphs admitting a perfect matching. A graph G is *factor-critical* if $G - u$ has a perfect matching for every vertex u of G.

The following facts are frequently used in our arguments.

Lemma 1 ([21]). *Every connected claw-free even graph admits a perfect matching.*

Corollary 2 ([21]). *Every 2-connected claw-free odd graph is factor-critical.*

Lemma 3 ([22]). *A connected graph is randomly matchable if and only if it is isomorphic to K_{2p} or $K_{p,p}$ ($p \geq 1$).*

Using the above facts, we identify some easy cases as follows.

Proposition 4. *A connected even graph is claw-free and equimatchable if and only if it is isomorphic to K_{2p} ($p \geq 1$) or C_4.*

Proof. The graphs K_{2p} and C_4 are clearly equimatchable and claw-free. Conversely, let G be a connected equimatchable claw-free even graph. By Lemma 1, G admits a perfect matching. Therefore, G is randomly matchable. By Lemma 3, G is either a $K_{p,p}$ or a K_{2p} for some $p \geq 1$. Since G is a claw-free graph, it is a K_{2p} or a C_4. □

Lemma 5. *Every odd graph G with $\alpha(G) = 2$ is equimatchable and claw-free.*

Proof. Every matching of G has at most $(n-1)/2$ edges since n is odd. On the other hand, a maximal matching with less than $(n-1)/2$ edges implies an independent set with at least 3 vertices, a contradiction. Then every maximal matching has exactly $(n-1)/2$ edges. The graph G is clearly claw-free because a claw contains an independent set with 3 vertices. □

Thus, from here onwards, we focus on the case where G is odd and $\alpha(G) \geq 3$. The following lemmas provide the main tools to obtain our characterization in Section 3 and enable us to confine the rest of this study to the cases with connectivity at most 3.

Lemma 6. *Let G be a connected equimatchable claw-free odd graph and M be a matching of G. Then the following hold:*

(i) *Every maximal matching of G leaves exactly one vertex exposed.*
(ii) *The subgraph $G \setminus V(M)$ contains exactly one odd connected component and this component is equimatchable.*
(iii) *The even connected components of $G \setminus V(M)$ are randomly matchable.*

Proof.

(i) Let v be a non-cut vertex of G (every graph has such a vertex). Then $G - v$ is a connected claw-free even graph, which by Lemma 1 admits a perfect matching with size $(n-1)/2$. This matching is clearly a maximum matching of G that leaves exactly one vertex exposed. Since G is equimatchable, every maximal matching of G leaves exactly one vertex exposed.

(ii) Since G is odd and $V(M)$ has an even number of vertices, $G \setminus V(M)$ contains at least one odd component. If $G \setminus V(M)$ contains two odd components, then every maximal matching extending M leaves at least two exposed vertices, contradicting (i). Let G_1 be the unique odd component of $G \setminus V(M)$. Assume for a contradiction that some maximal matching M_1 of G_1 leaves at least three exposed vertices. Then any maximal matching of G extending $M \cup M_1$ leaves at least three exposed vertices, contradicting (i). Therefore, every maximal matching of G_1 leaves exactly one vertex exposed; i.e., G_1 is equimatchable.

(iii) Let G_i be an even component of $G \setminus V(M)$. Assume for a contradiction that there is a maximal matching M_i of G_i leaving at least two exposed vertices. Then any maximal matching of G extending $M \cup M_i$ leaves at least two exposed vertices, contradicting (i). □

Lemma 7. *Let G be a connected claw-free odd graph. Then G is equimatchable if and only if for every independent set I of 3 vertices, $G \setminus I$ has at least two odd connected components.*

Proof. As in the proof of Lemma 6(i), picking up a non-cut vertex v of G, the perfect matching of $G - v$ is a matching of G with $(n-1)/2$ edges.

($\Rightarrow$) Assume that G is equimatchable, and let I be an independent set of G with 3 vertices. Suppose, for a contradiction, that all connected components of $G \setminus I$ are even. Thus, every such connected component admits a perfect matching by Lemma 1. The union of all these matchings is a maximal matching of G with size $(n-3)/2$, contradicting the equimatchability of G. Then $G \setminus I$ has at least one odd component. The claim follows from parity considerations.

($\Leftarrow$) Assume that G is not equimatchable. Then G has a maximal matching M of size $(n-3)/2$ by the following fact. Consider any maximal matching M' of G with size $(n-\ell)/2$ for some $\ell \geq 3$. If $\ell \geq 4$ find an M'-augmenting path and increase M' along this augmenting path. Indeed, the new matching M'' obtained in this way is still maximal (the set of vertices exposed by M'' is a subset of vertices exposed by M') and contains one more edge. We repeat this procedure until the matching reaches size $(n-3)/2$. Then $I = G \setminus V(M)$ is an independent set with size 3 and $G \setminus I$ has a perfect matching, namely M. This implies that every connected component of $G \setminus I$ is even. □

Corollary 8. *If G is an equimatchable claw-free odd graph with $\alpha(G) \geq 3$, then $\kappa(G) \leq 3$.*

Proof. Let I be an independent set of G with three vertices, and assume for a contradiction that $\kappa(G) \geq 4$. Then $G \setminus I$ is connected and even, contradicting Lemma 7. □

3. Equimatchable claw-free odd graphs with $\alpha(G) \geq 3$

Let G be a connected equimatchable claw-free odd graph with $\alpha(G) \geq 3$. By Corollary 8, $\kappa(G) \leq 3$. Since $\alpha(G) \geq 3$, G contains independent sets I of three vertices, each of which is a 3-cut by Lemma 7. If $\kappa(G) = 3$, then every such I is a minimal cut set. In Section 3.1 (see Lemma 14) we show that the other direction also holds; i.e. if every such I is a minimal cut set, then $\kappa(G) = 3$. Therefore, if $\kappa(G) = 2$, at least one independent 3-cut I is not minimal; i.e. G contains two non-adjacent vertices forming a cut set (we will call this cut set a strongly independent 2-cut). We analyze this case in Section 3.2. Finally, we analyze the case $\kappa(G) = 1$ in Section 3.3.

In each subsection we describe the related graph families. Although we will use their full descriptions in the proofs, we also introduce the following notation for a more compact description that will be useful in the illustrations of Fig. 2 and in the recognition algorithm given in Corollary 27. Let H be a graph on k vertices $v_1, v_2, \ldots, v_k$ and let $n_1, n_2, \ldots, n_k$ be non-negative integers denoting the *multiplicities* of the corresponding vertices. Then $H(n_1, n_2, \ldots, n_k)$ denotes the graph obtained from H by repeatedly replacing each vertex v_i with a clique of $n_i \geq 0$ vertices, each of which having the same neighborhood as v_i; i.e. each vertex in such a clique is a *twin* of v_i. Clearly, $H = H(1, \ldots, 1)$ where all multiplicities are 1. Note that if $n_i = 0$ for some i, this means that the vertex v_i is deleted.

The following observations will be useful in our proofs.

Lemma 9. *Let G be a connected claw-free graph, S be a minimal cut set of G, C be an induced cycle of $G \setminus S$ with at least 4 vertices, and K be a clique of $G \setminus S$. Then*

(i) *$G \setminus S$ consists of exactly two connected components, and every vertex of S has a neighbor in both of them.*
(ii) *The set $N_{G_i}(s)$ is a clique for every vertex $s \in S$ and every connected component G_i of $G \setminus S$.*
(iii) *The neighborhood of every vertex of S in C is either empty or consists of exactly two adjacent vertices of C.*
(iv) *If s_1 and s_2 are two non-adjacent vertices of S, then $N_K(s_1) \cap N_K(s_2) = \emptyset$ or $N_K(s_1) \cup N_K(s_2) = K$.*

Proof.

(i) By the minimality of S, every vertex $s \in S$ is adjacent to at least two components of $G \setminus S$. Assume for a contradiction that a vertex $s \in S$ is adjacent to three connected components of $G \setminus S$. Then, s together with one arbitrary vertex adjacent to it from each component form a claw, contradiction. Therefore, every vertex $s \in S$ is adjacent to exactly two components of $G \setminus S$. Furthermore, by the minimality of S, every component is adjacent to every vertex of S. Therefore $G \setminus S$ consists of exactly two connected components.

(ii) Let $s \in S$, and G_1, G_2 be the two connected components of $G \setminus S$. Assume that the claim is not correct. Then, without loss of generality, there are two non-adjacent vertices $w, w' \in N_{G_1}[s]$. Then s, w, w' together with an arbitrary vertex of $N_{G_2}(s)$ form a claw, contradiction.

(iii) Let $s \in S$ be adjacent to a vertex v of C. If s is adjacent to none of the two neighbors of v in C, then v, s, and the two neighbors v in C form a claw, contradiction. If s has three neighbors in C, then its neighborhood in the connected component of C is not a clique, contradicting (ii).

(iv) Assume for a contradiction that $N_K(s_1) \cap N_K(s_2) \neq \emptyset$ and $N_K(s_1) \cup N_K(s_2) \subset K$. Let $c \in N_K(s_1) \cap N_K(s_2)$ and $a \in K \setminus N_K(s_1) \cup N_K(s_2)$. Then $\{s_1, s_2, a, c\}$ induces a claw, contradiction. $\square$

3.1. *Equimatchable claw-free odd graphs with $\alpha(G) \geq 3$ and $\kappa(G) = 3$*

In this section we show that the class of claw-free equimatchable odd graphs with independence number at least 3 and connectivity 3 is equivalent to the following graph class.

Definition 1. Graph $G \in \mathcal{G}_3$ if it has an independent 3-cut $S = \{s_1, s_2, s_3\}$ such that

(i) The subgraph $G \setminus S$ consists of two connected components A and A', each of which is an odd clique of at least three vertices

(ii) there exist two vertices $a \in A$, $a' \in A'$ such that

- $N(s_1) = A + a'$,
- $N(s_2) = A' + a$, and
- $N(s_3) = A \cup A' \setminus \left\{a, a'\right\}$.

We note that

$$\mathcal{G}_3 = \{G_3(1, 2p, 1, 1, 1, 2q, 1) | \ p, q \geq 1\}$$

where G_3 is the graph depicted in Fig. 2g.

Proposition 10. *If $G \in \mathcal{G}_3$, then G is a connected equimatchable claw-free odd graph with $\alpha(G) = \kappa(G) = 3$.*

Proof. The only independent sets of G_3 with 3 vertices are $S = \{s_1, s_2, s_3\}$ and $S' = \left\{s_3, a, a'\right\}$. Both $G \setminus S$ and $G \setminus S'$ have two odd components; hence, G is equimatchable by Lemma 7. All other properties are easily verifiable. $\square$

The following lemma provides the general structure of the claw-free equimatchable odd graphs with $\alpha(G) \geq 3$ and $\kappa(G) \leq 3$.

Lemma 11. *Let G be an equimatchable claw-free odd graph. If $S = \{s_1, s_2, s_3\}$ is a minimal independent cut set of G, then $G \setminus S$ consists of two odd cliques A and A', each of which has at least three vertices, and every vertex of S has a neighbor in both A and A'.*

Proof. By Lemma 9(i), $G \setminus S$ consists of two components A and A'. By Lemma 7, both A and A' are odd. If one of A of A' consists of a single vertex, then this single vertex together with S forms a claw. Therefore, each of A and A' has at least three vertices. It remains to show that both A and A' are cliques.

Let v, v' be two vertices of $G[A]$ such that the distance between v and v' is as large as possible. If v and v' are adjacent, then A is a clique. Now suppose that $vv' \notin E(G)$. We claim that neither v nor v' is a cut vertex of $G[A]$. Suppose that $G[A \setminus v]$ has two connected components, B and B'. Without loss of generality, let v' be in B'. Then every vertex b in B is further from v' than v is, since every path between b and v' contains v, a contradiction. Therefore, neither v nor v' is a cut vertex of $G[A]$, as claimed. At least one of v, v' is adjacent to at most one vertex of S because otherwise, by counting arguments, at least one vertex of S is adjacent to both v and v', contradicting Lemma 9(ii). Assume without loss of generality that v is non-adjacent to $\{s_1, s_2\}$, and consider the independent set $I = \{s_1, s_2, v\}$. If v is not the unique vertex of A adjacent to s_3, then $G \setminus I$ is connected and even, contradicting Lemma 7. Otherwise, $G \setminus I$ consists of two even components, again contradicting Lemma 7. Therefore, A is a clique, and by symmetry, so is A'. $\square$

We note that Lemma 11 is a variant of the following result in the literature for the case $k = 3$; indeed Lemma 11 is also valid for connectivity 1 and 2. This will enable us to replace the connectivity 3 condition with the existence of a minimal independent cut set of three vertices in what follows.

Lemma 12 ([3]). *Let G be a k-connected equimatchable factor-critical graph with at least $2k+3$ vertices and a k-cut S such that $G \setminus S$ has two components with at least 3 vertices, where $k \geq 3$. Then $G \setminus S$ has exactly two components and both are complete graphs.*

Proposition 13. *If G is an equimatchable claw-free odd graph with $\alpha(G) \geq 3$ and it contains a minimal independent cut set $S = \{s_1, s_2, s_3\}$ with three vertices, then $G \in \mathcal{G}_3$.*

Proof. By Lemma 11, Property (i) of Definition 1 holds. We proceed to show (ii). Since S is minimal, every vertex $s \in S$ has a neighbor in each of A and A'. Suppose that a connected component of $G \setminus S$, say A, has a vertex v that is non-adjacent to two vertices, say s_1, s_2 of S. Then $I = \{s_1, s_2, v\}$ is an independent set with three vertices and $G \setminus I$ is either connected, or has two even components $A' + s_3$ and $A - v$ (when $N_A(s_3) = \{v\}$), contradicting Lemma 7. Therefore, every vertex of $A \cup A'$ is adjacent to at least two vertices of S. As already observed, a vertex of $A \cup A'$ that is complete to S implies a claw, contradiction. We conclude that every vertex of $A \cup A'$ is adjacent to exactly two vertices of S. For $i, j \in [3]$, let $N_{i,j} = N_A(s_i) \cap N_A(s_j)$ and $N'_{i,j} = N_{A'}(s_i) \cap N_{A'}(s_j)$. We have shown that $\{N_{1,2}, N_{2,3}, N_{1,3}\}$ (resp. $\{N'_{1,2}, N'_{2,3}, N'_{1,3}\}$) is a partition of A (resp. A').

Assume that for some pair (i, j) none of $N_{i,j}, N'_{i,j}$ is empty, and let $k = 6 - i - j$. Consider the set $S' = \{s_k, w_{ij}, w'_{ij}\}$ where w_{ij} and w'_{ij} are arbitrary vertices of $N_{i,j}$ and $N'_{i,j}$, respectively. S' is an independent set, and it is easy to verify that if one of $N_{i,j}$ and $N'_{i,j}$ is not a singleton, say $N_{i,j}$, then $G \setminus S'$ is connected; indeed, in this case, there exists a vertex $u_{ij} \in N_{i,j} - w_{ij}$. Moreover, either $N'_{i,k}$ or $N'_{j,k}$ is non-empty (since otherwise s_k would not have a neighbor in A', contradicting the minimality of S) implying that $G \setminus S'$ is connected. Therefore, for every pair (i, j) either one of $N_{i,j}, N'_{i,j}$ is empty or both are singletons.

Suppose that for every pair (i, j) one of $N_{i,j}, N'_{i,j}$ is empty. Then at least 3 of the 6 sets are empty, and two of them must be in the same component, say A. Suppose that, for instance $N_{1,2} = N_{1,3} = \emptyset$. Then $N_A(s_1) = N_{1,2} \cup N_{1,3} = \emptyset$, a contradiction. Therefore, for at least one pair (i, j), both $N_{i,j}$ and $N'_{i,j}$ are singletons. We can renumber the vertices of S such that $N_{1,2} = \{a\}$ and $N'_{1,2} = \{a'\}$ are singletons. Now suppose that for some other pair, say $(2, 3)$, $N_{2,3} = \{w_{23}\}$ and $N'_{2,3} = \{w'_{23}\}$ are singletons. Then the matching $\{a'w'_{23}, s_1a, s_3w_{23}\}$ disconnects G into three odd components, contradicting Lemma 6(ii). Therefore, both pairs $(2, 3)$ and $(1, 3)$ fall into the other category, i.e. one of $N_{2,3}, N'_{2,3}$ and one of $N_{1,3}, N'_{1,3}$ is empty. Since two sets from the same component cannot be empty, we conclude that without loss of generality $N_{2,3} = N'_{1,3} = \emptyset$. In other words, $A = N_{1,3} + a$ and $A' = N'_{2,3} + a'$. Hence, property (ii) also holds. □

We conclude this section with the following summarizing lemma.

Lemma 14. *Let G be an equimatchable claw-free odd graph with $\alpha(G) \geq 3$. Then the following conditions are equivalent:*

(i) $\kappa(G) = 3$,
(ii) *every independent set S of G with three vertices is a minimal cut set,*
(iii) $G \in \mathcal{G}_3$.

Proof. (i) $\Rightarrow$ (ii) Let S be an independent set with three vertices. By Lemma 7, S is a cut set, and since $\kappa(G) = 3$ it is a minimal cut set.

(ii) $\Rightarrow$ (iii) By Proposition 13.

(iii) $\Rightarrow$ (i) We observe that G_3 contains only one 2-cut set, namely $\{v_2, v_6\}$. Since the multiplicities of v_2 and v_6 are at least two, this set does not yield a 2-cut of G. □

3.2. Equimatchable claw-free odd graphs with $\alpha(G) \geq 3$ and $\kappa(G) = 2$

Throughout this section, G is an equimatchable claw-free odd graph with $\alpha(G) \geq 3$ and $\kappa(G) = 2$, I is an independent set with three vertices, and $S = \{s_1, s_2\}$ is a (minimal) cut set of G. Recall that, by Corollary 2, G is factor-critical, and note that since G is connected and $\alpha(G) \geq 3$, we have $n \geq 4$. Our starting point is the following result on 2-connected equimatchable factor-critical graphs.

Lemma 15 ([5]). *Let G be a 2-connected, equimatchable factor-critical graph with at least 4 vertices and $S = \{s_1, s_2\}$ be a minimal cut set of G. Then $G \setminus S$ has precisely two components, one of them even and the other odd. Let A_S and B_S denote the even and odd components of $G \setminus S$, respectively. Let a_1 and a_2 be two distinct vertices of A_S adjacent to s_1 and s_2, respectively, and, if $|B_S| > 1$, let b_1 and b_2 be two distinct vertices of B_S adjacent to s_1 and s_2, respectively. Then the following hold:*

1. *The subgraph B_S is one of the four graphs K_{2p+1}, $K_{2p+1} - b_1b_2$, $K_{p,p+1}$, $K_{p,p+1} + b_1b_2$ for some $p \geq 1$. In the last two cases, all neighbors of S in B_S belong to the larger part of the bipartition of $K_{p,p+1}$.*
2. *The subgraph $A_S \setminus \{a_1, a_2\}$ is connected and randomly matchable, and if $|B_S| > 1$, then A_S is connected and randomly matchable.*

In the rest of this section, A_S, B_S denote the even and odd connected components of $G \setminus S$, respectively, and $a_1, a_2 \in A_S$ and $b_1, b_2 \in B_S$ are as described in Lemma 15. We note that the vertices a_1, a_2 exist, since otherwise A_S contains a cut vertex of G.

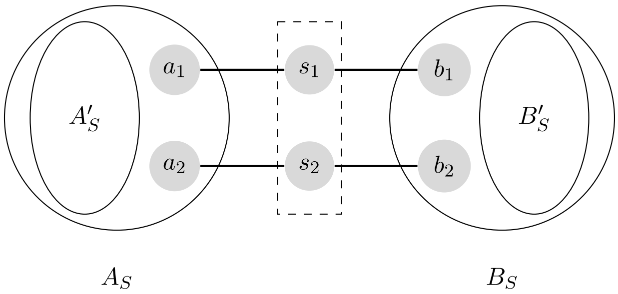


Fig. 1. The structure of 2-connected equimatchable claw-free odd graphs by Lemma 15.

Similarly, if $|B_S| \geq 3$, the vertices b_1 and b_2 exist. Moreover, let $A'_S = A_S \setminus \{a_1, a_2\}$, and $B'_S = B_S \setminus \{b_1, b_2\}$ whenever $|B_S| > 1$ (see Fig. 1). The minimal cut set S is *independent* if $s_1s_2 \notin E(G)$, and *strongly independent* if there exists an independent set I with three vertices including S.

An important consequence of Section 3.1 which will guide our proofs is the following:

Corollary 16 (*of Lemma 14*). *Let G be an equimatchable claw-free odd graph with $\alpha(G) \geq 3$. If $\kappa(G) = 2$, then it has a strongly independent 2-cut.*

Proof. Since $\kappa(G) = 2$, by Lemma 14, there exists an independent 3-cut I that is not a minimal cut set, i.e. I contains a minimal 2-cut $S \subseteq I$. Moreover, since $S \subseteq I$, S is strongly independent. □

The main result of this section is that G is either a C_7 or in one of the following graph families. The reader is referred to Fig. 2d–f for these definitions.

Definition 2. A graph is in $\mathcal{G}_{21}$ if its vertex set can be partitioned into V_1 and V_2 such that

(i) V_1 induces a K_{2q+1} for some $q \geq 1$,
(ii) V_2 induces a C_4, say $v_1v_2v_3v_4$,
(iii) $N_{V_1}(v_1) = N_{V_1}(v_2)$,
(iv) $2 \leq \left|N_{V_1}(v_1)\right| < |V_1|$, and
(v) $N_{V_1}(v_3) = N_{V_1}(v_4) = \emptyset$.

A graph is in $\mathcal{G}_{22}$ if it has an independent 2-cut $S = \{s_1, s_2\}$ such that

(i) A_S is a K_{2p} for some $p \geq 1$,
(ii) B_S is a K_{2q+1} for some $q \geq 0$,
(iii) s_1 and s_2 are complete to B_S,
(iv) $N_{A_S}(s_1) \cup N_{A_S}(s_2) \subsetneq A_S$, and
(v) $N_{A_S}(s_1) \cap N_{A_S}(s_2) = \emptyset$.

A graph is in $\mathcal{G}_{23}$ if it has an independent 2-cut $S = \{s_1, s_2\}$ such that

(i) A_S is a K_2,
(ii) $G[S \cup A_S]$ is a P_4,
(iii) B_S is a K_{2q+1} for some $q \geq 1$, and
(iv) $N_{B_S}(s_1) \cup N_{B_S}(s_2) = B_S$, $N_{B_S}(s_1) \neq \emptyset$, $N_{B_S}(s_2) \neq \emptyset$, either $N_{B_S}(s_1) \neq B_S$ or $N_{B_S}(s_2) \neq B_S$.

We note that

$$\begin{aligned}
\mathcal{G}_{21} &= \{G_{21}(1, 1, 1, 1, x, 2q+1-x) \mid 2 \leq x \leq 2q\},\\
\mathcal{G}_{22} &= \{G_{22}(2p-x-y, x, y, 1, 1, 2q+1) \mid q \geq 0, x, y \geq 1, x+y \leq 2p-1\},\\
\mathcal{G}_{23} &= \{G_{23}(1, 1, 1, 1, x, y, 2q+1-x-y) \mid 1 \leq x+y \leq 2q+1\}
\end{aligned}$$

where G_{21}, G_{22}, G_{23} are the graphs depicted in Fig. 2d, e and f, respectively. It can be noticed that the vertices s_1 and s_2 are not identified in G_{21} of Fig. 2e since the vertices playing the roles of s_1 and s_2 will depend on the case under analysis for this family.

Proposition 17. *If $G \in \mathcal{G}_{21} \cup \mathcal{G}_{22} \cup \mathcal{G}_{23} + C_7$, then G is a connected equimatchable claw-free odd graph with $\alpha(G) \geq 3$ and $\kappa(G) = 2$.*

Proof. All the other properties being easily verifiable, we will check the equimatchability of a graph $G \in \mathcal{G}_{21} \cup \mathcal{G}_{22} \cup \mathcal{G}_{23} + C_7$ by using Lemma 7. One can observe that in each case, there is only one possible type (up to isomorphisms) of independent set I of three vertices which is as described below.

If $G \in \mathcal{G}_{21}$ then I consists of v_1, v_3 and a vertex in V_1. Then $G \setminus I$ consists of one component with the single vertex v_4 and the other $G \setminus (I + v_4)$ which is odd.

If $G \in \mathcal{G}_{22}$ then I consists of s_1, s_2 and a vertex $a \in A_S \setminus (N_{A_S}(s_1) \cup N_{A_S}(s_2))$. Then $G \setminus I$ consists of two odd components, namely B_S and $A_S - a$.

If $G \in \mathcal{G}_{23}$ then $I = \{a_1, s_2, b\}$ where $b \in B_S \setminus N_{B_S}(s_2)$ (assuming without loss of generality that $x > 0$). Then $G \setminus I$ consists of two odd components: the singleton $\{a_2\}$ and $G \setminus (I + a_2)$ which is odd.

Finally, if G is a C_7, then for any independent set I of three vertices, the graph $G \setminus I$ consists of two singletons and two adjacent vertices. □

In the rest of this section, we proceed as follows to prove the other direction: In Proposition 18, we analyze the case where A_S is a C_4 for some 2-cut S. Subsequently, in Observation 19 we summarize Lemma 15 for the case where A_S is not a C_4, and $|B_S| > 1$ where S is an independent 2-cut. We further separate this case into two. In Proposition 20, we give the exact structure of G when B_S is neither a singleton nor a P_3. In Proposition 21, we give the exact structure of G when B_S is a P_3. We complete the analysis in Proposition 22, which determines the exact structure of G in the last case, i.e. when A_S is not a C_4 and $|B_S| = 1$. In the proofs of Propositions 20–22, we heavily use the fact that the graph under consideration has a strongly independent 2-cut S. Moreover, this fact will allows us to conclude in Theorem 26 that we cover all possible cases for claw-free equimatchable odd graphs of connectivity 2.

Proposition 18. *If A_S is a C_4 for some 2-cut S of G, then $G \in \mathcal{G}_{21}$ and S is not independent.*

Proof. Let $S = \{s_1, s_2\}$ be a 2-cut of G, and A_S be a 4-cycle. In what follows, we show that $G \in \mathcal{G}_{21}$ by setting $V_2 = A_S$ and $V_1 = V(G) \setminus V_2 = S \cup B_S$. Since A_S is a 4-cycle, Property (ii) of $\mathcal{G}_{21}$ holds for G. Let A_S be the 4-cycle $v_1v_2v_3v_4$. By Lemma 9(iii), both $N_{A_S}(s_1)$ and $N_{A_S}(s_2)$ consist of two adjacent vertices of V_2. If $N_{V_2}(s_1) \neq N_{V_2}(s_2)$, then $N_{V_2}(s_1) \cup N_{V_2}(s_2)$ contains two non-adjacent vertices x, y such that $x \in N_{V_2}(s_1)$, and $y \in N_{V_2}(s_2)$. Then the matching $\{s_1x, s_2y\}$ isolates the two vertices of $V_2 \setminus \{x, y\}$, contradicting Lemma 6(ii). Therefore, $N_{V_2}(s_1) = N_{V_2}(s_2)$ and it consists of two adjacent vertices of V_2, say v_1 and v_2. Since S is a cut set, the neighbors of v_1 and v_2 in V_1 are exactly s_1 and s_2, thus showing (iii) and the first inequality of (iv). The second part of the inequality follows from the fact that S is a cut-set and $V_1 \setminus S = B_S \neq \emptyset$. Furthermore, (v) holds since $S \subsetneq V_1$ is a cut set and the neighborhood of S in V_2 consists of $\{v_1, v_2\}$.

It remains to show Property (i), i.e. that V_1 is an odd clique. Observe that $s_1s_2 \in E(G)$ since otherwise $S + v_2 + v_3$ forms a claw. Thus, S is not independent. The matching $\{v_1v_4, v_2s_2\}$ leaves the singleton v_3 as an odd component. Therefore, by Lemma 6(ii) and (iii), $G[V_1 - s_2]$ is randomly matchable which is either an even clique or a C_4 by Proposition 4. Suppose that $G[V_1 - s_2]$ is the cycle $s_1b_1bb_3$. We have that $N_{B_S}(s_1) = \{b_1, b_3\}$ is not a clique, contradicting Lemma 9(ii). Therefore, $G[V_1 - s_2]$ is a K_{2q} for some $q \geq 1$. By symmetry, $G[V_1 - s_1]$ is also a K_{2q}. Since $s_1s_2 \in E(G)$, we conclude that $G[V_1]$ is a K_{2q+1} for some $q \geq 1$. □

Observation 19. *If S is an independent 2-cut of G and $|B_S| > 1$, then*

(i) *The subgraph $G[A_S]$ is a K_{2p} for some $p \geq 1$, and*
(ii) *The subgraph $G[B_S]$ is either a K_{2q+1}, or $K_{2q+1} - b_1b_2$ for some $q \geq 1$.*

Proof.

(i) By Lemma 15, A_S is connected and randomly matchable. By Proposition 18, A_S is not a C_4. Then, by Proposition 4, A_S is a K_{2p} for some $p \geq 1$.
(ii) Recall Lemma 15. In this case, B_S cannot be a $K_{q,q+1}$ or $K_{q,q+1} + b_1b_2$ for $q \geq 2$ since otherwise (recalling that b_1 is in the larger part of the bipartition) s_1, b_1 and two vertices adjacent to b_1 in the smaller part of the bipartition of B_S induce a claw. For $q = 1$ we note that $K_{1,2} = K_3 - e$ and $K_{1,2} + e = K_3$. Therefore, B_S is either a K_{2q+1}, or a $K_{2q+1} - b_1b_2$ for some $q \geq 1$. □

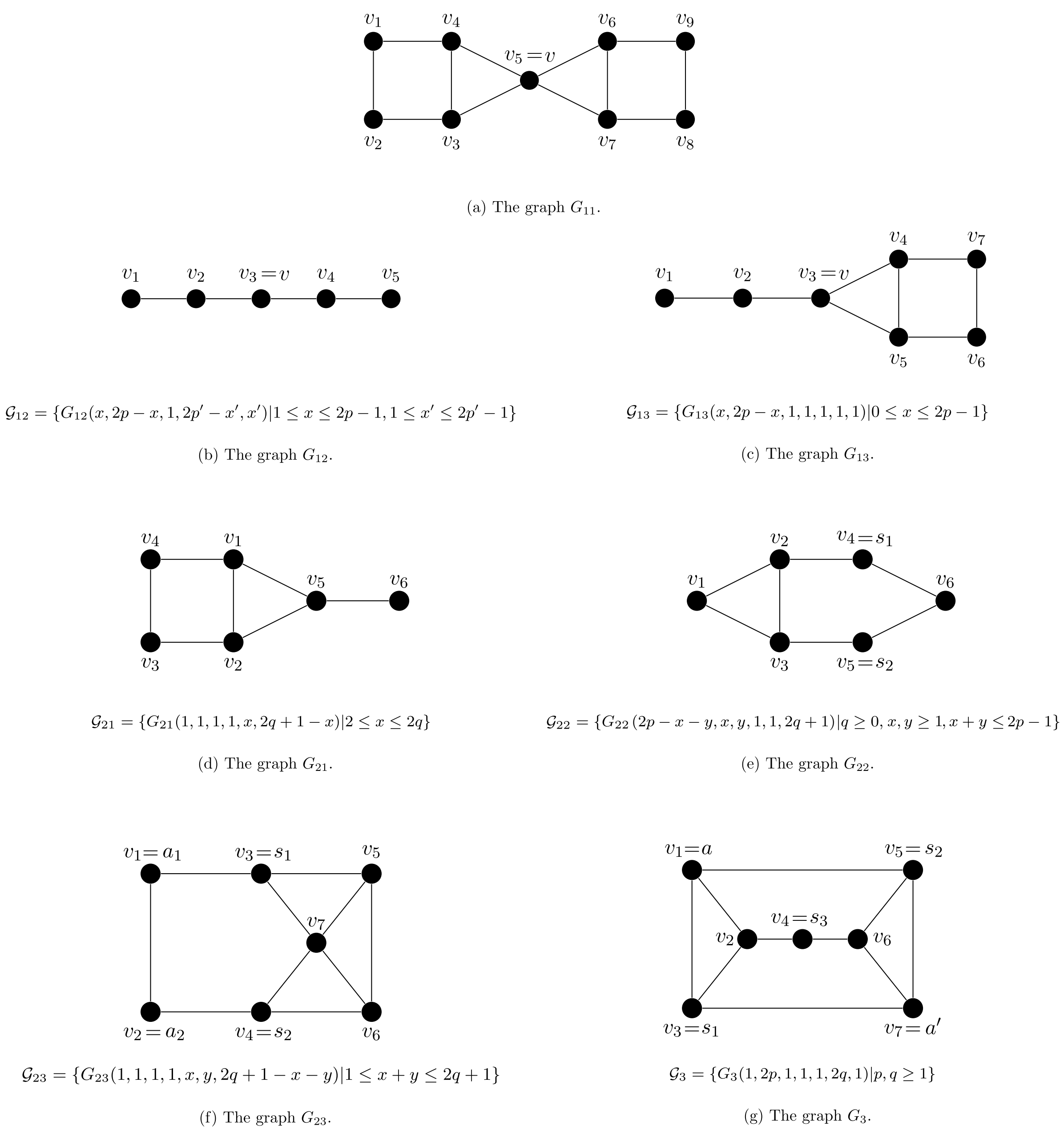
Proposition 20. *If there exists a strongly independent 2-cut S of G such that $|B_S| > 1$ and B_S is not a P_3, then $G \in \mathcal{G}_{22}$.*

Proof. We now show that G has all the properties of $\mathcal{G}_{22}$. Let $S = \{s_1, s_2\}$ be a strongly independent 2-cut of G, and I be an independent set of three vertices containing S.

The fact that A_S is a K_{2p} for some $p \geq 1$ (Property (i)) follows from Observation 19.

By the same observation and since B_S is not a P_3, B_S is either a K_{2q+1} for some $q \geq 1$ or a $K_{2q+1} - b_1b_2$ for some $q \geq 2$, thus 2-connected. Note that the unique vertex in $I \setminus S$ is not in B_S, since otherwise $G \setminus I$ consists of two even components. Therefore $I = S + a$ for some $a \in A_S$. This implies that $N_{A_S}(s_1) \cup N_{A_S}(s_2) \subsetneq A_S$, thus Property (iv) is verified.

Now suppose that there exists a vertex $b \in B_S$ that is non-adjacent to s_1. Since both A_S and B_S are 2-connected, both of $A_S - a$ and $B_S - b$ are connected. Moreover, s_2 is adjacent to $A_S - a$. Then $I' = \{a, s_1, b\}$ is an independent set such that $G \setminus I'$



(a) The graph G_{11}.

$\mathcal{G}_{12} = \{G_{12}(x, 2p - x, 1, 2p' - x', x') | 1 \le x \le 2p - 1, 1 \le x' \le 2p' - 1\}$

(b) The graph G_{12}.

$\mathcal{G}_{13} = \{G_{13}(x, 2p - x, 1, 1, 1, 1, 1) | 0 \le x \le 2p - 1\}$

(c) The graph G_{13}.

$\mathcal{G}_{21} = \{G_{21}(1, 1, 1, 1, x, 2q + 1 - x) | 2 \le x \le 2q\}$

(d) The graph G_{21}.

$\mathcal{G}_{22} = \{G_{22}(2p - x - y, x, y, 1, 1, 2q + 1) | q \ge 0, x, y \ge 1, x + y \le 2p - 1\}$

(e) The graph G_{22}.

$\mathcal{G}_{23} = \{G_{23}(1, 1, 1, 1, x, y, 2q + 1 - x - y) | 1 \le x + y \le 2q + 1\}$

(f) The graph G_{23}.

$\mathcal{G}_{3} = \{G_{3}(1, 2p, 1, 1, 1, 2q, 1) | p, q \ge 1\}$

(g) The graph G_3.

Fig. 2. The graph G_{11}.

is either connected or consists of two even components $A_S - a + s_2$ and $B_S - b$, contradicting Lemma 7. We conclude that s_1, and by symmetry s_2, are complete to B_S. This proves Property (iii).

Since s_1 is complete to B_S, B_S is a clique by Lemma 9(ii). This shows Property (ii).

Finally, Property (v) follows from Property (iv) and Lemma 9(iv). $\square$

Proposition 21. *If there exists some strongly independent 2-cut S of G such that B_S is a P_3, then $G \in \mathcal{G}_{23} + C_7$.*

Proof. Let $S = \{s_1, s_2\}$ be a strongly independent 2-cut of G. We now show that G is either a C_7 or has the following properties:

(i) The subgraph A_S is a K_{2p} for some $p \ge 1$,
(ii) The subgraph $G[S \cup B_S]$ is a P_5,
(iii) $N_{A_S}(s_1) = N_{A_S}(s_2) = A_S$.

Then, we will show that these properties imply that $G \in \mathcal{G}_{23}$ with zero copies of v_7 and an independent 2-cut different from S.

(i) Follows from Observation 19.

(ii) The subgraph $G[B_S]$ is a path $b'_1 b' b'_2$. If s_1 is not adjacent to any of b'_1 and b'_2 then s_1 is adjacent to b' and $B + s_1$ is a claw. Therefore, without loss of generality s_1 is adjacent b'_1. If $s_1 b'_2 \in E(G)$, then $N_B(s_1)$ is not a clique, contradicting Lemma 9(ii). Therefore, $s_1 b'_2 \notin E(G)$. Let a be an arbitrary element of $A_S - a_2$. Clearly, $A_S - a + s_2$ is connected. If $s_1 b' \in E(G)$ then $I = \{a, b'_1, b'_2\}$ is an independent set such that $G \setminus I$ is either connected or has two even components. Therefore, $s_1 b' \notin E(G)$, concluding that $N_{B_S}(s_1) = \{b'_1\}$. Symmetrically, we have $N_{B_S}(s_2) = \{b'_2\}$. Therefore, $b'_1 = b_1$ and $b'_2 = b_2$, thus $S \cup B_S$ induces the $P_5 = s_1 b_1 b b_2 s_2$.

(iii) Recall that $A'_S = A_S - a_1 - a_2$. First assume that $A'_S \neq \emptyset$. Furthermore, suppose that there is some $a' \in A'_S$ not adjacent to s_1. Then $I' = \{s_1, a', b_2\}$ is an independent set and $G \setminus I'$ has two even components, contradicting Lemma 7. Therefore, s_1 is complete to A'_S and symmetrically so is s_2. Now suppose that $s_1 a_2 \notin E(G)$, and consider the independent set $I'' = \{s_1, a_2, b_2\}$. Then, $G \setminus I''$ has two even components, contradicting Lemma 7. Therefore, $s_1 a_2 \in E(G)$, and symmetrically $s_2 a_1 \in E(G)$. We conclude that $N_{A_S}(s_1) = N_{A_S}(s_2) = A_S$. Now assume that $A'_S = \emptyset$, i.e. $A_S = \{a_1, a_2\}$. Then $a_1 a_2 s_2 b_2 b b_1 s_1$ is a Hamiltonian cycle of G. The edge set of G possibly contains one or both of the edges $a_1 s_2$, $a_2 s_1$. If both are edges of G, then $N_{A_S}(s_1) = N_{A_S}(s_2) = A_S$ and we are done. If none is an edge of G, then G is a C_7. We remain with the case that exactly one of $a_1 s_2$, $a_2 s_1$, say $a_1 s_2$ is an edge of G. In this case $\{a_2, s_1, b_2\}$ is an independent set whose removal separates G into two even components, contradicting Lemma 7.

We now observe that the above properties imply $G \in \mathcal{G}_{23}$. Indeed, let S' be the independent set $\{s_1, b_2\}$, and verify the properties of $\mathcal{G}_{23}$: (i) $A_{S'} = \{b, b_1\}$ is a K_2, (ii) $G[S' \cup A_{S'}] = G[\{s_1, b_2, b, b_1\}]$ is the $P_4 = s_1 b_1 b b_2$, (iii) $B_{S'} = A_S + s_2$ is an odd clique since A_S is an even clique and s_2 is complete to it, (iv) s_1 is complete to A_S and b_2 is adjacent to s_2, thus $N_{B_{S'}}(s_1) \cup N_{B_{S'}}(b_2) = B_{S'}$ and $N_{B_{S'}}(s_1), N_{B_{S'}}(b_2) \neq \emptyset$, furthermore $N_{B_{S'}}(s_1) \neq B_{S'}$ since $s_1 s_2 \notin E(G)$. □

Proposition 22. *If for every 2-cut S of G the component A_S is not a C_4, and for every strongly independent 2-cut S of G the component B_S consists of a single vertex, then $G \in \mathcal{G}_{21} \cup \mathcal{G}_{22} \cup \mathcal{G}_{23}$.*

Proof. Let $S = \{s_1, s_2\}$ be a strongly independent 2-cut of G. We remark that in this case we cannot use Observation 19. Moreover, the only fact that we can deduce from Lemma 15 is that A'_S is randomly matchable, a fact that is easily observed by applying Lemma 6 to the matching $\{s_1 a_1, s_2 a_2\}$.

We first observe that there are no 2-connected claw-free graphs on at most 5 vertices with an independent set of three vertices. Therefore, we can assume that $|V(G)| > 5$, i.e. that $A'_S \neq \emptyset$.

We proceed with the proof by considering two disjoint cases.

- $N_{A'_S}(s_1) = N_{A'_S}(s_2) = \emptyset$: In this case we will show that G has all the properties of $\mathcal{G}_{22}$. Properties (ii), (iii), (iv) clearly hold for G. If A'_S is a C_4, then $S' = \{a_1, a_2\}$ is a 2-cut with $A_{S'}$ being a C_4, contradicting our assumptions. Therefore, A'_S is a K_{2p} for some $p \geq 1$. If $a_1 s_2 \in E(G)$, then s_1, s_2, a_1 and any neighbor of a_1 in A'_S induce a claw, contradiction. Therefore, and using symmetry, we have that $a_1 s_2, a_2 s_1 \notin E(G)$, i.e. Property (v) holds. It remains to show that A_S is a clique.
  If $a_1 a_2 \notin E(G)$, then $S' = \{a_1, a_2\}$ is a strongly independent cut with $B_{S'}$ being a P_3, contradicting our assumptions. Therefore, $a_1 a_2 \in E(G)$. We now show that A_S is a clique by proving that a_1 is complete to A'_S, and so is a_2 by symmetry. We first observe that $N_{A'_S}(a_1) \subseteq N_{A'_S}(a_2)$. Indeed, otherwise there is a vertex $a' \in A'_S$ adjacent to a_1 and not adjacent to a_2, and $\{a_1, a_2, s_1, a'\}$ induces a claw. By symmetry, we get $N_{A'_S}(a_1) = N_{A'_S}(a_2)$. This neighborhood has at least two vertices since otherwise $\kappa(G) = 1$ where the unique common neighbor of a_1 and a_2 is a cut vertex. Now, suppose that a_1 is not complete to A'_S and let $a' \in A'_S$ be non-adjacent to a_1. Then $I' = \{a', a_1, s_2\}$ is an independent set. Furthermore, $G \setminus I'$ consists of two even components, a contradiction to Lemma 7. Therefore, a_1 is complete to A'_S, and so is a_2 by symmetry.
- $N_{A'_S}(s_1) \neq \emptyset$: We start by showing that $A_1 = A'_S + a_1$ is a clique. Let $a'_1 \in N_{A'_S}(s_1)$ and apply Lemma 6 to the matching $\{s_1 a'_1, s_2 a_2\}$. It implies that $G[A'_S + a_1 - a'_1]$ is randomly matchable. Suppose that $G[A'_S + a_1 - a'_1]$ is a $C_4 = a_1 a'_2 a'_3 a'_4$. Then $a_1 a'_3, a'_2 a'_4 \notin E(G)$. Then A'_S is not a clique, thus it is the $C_4 = a'_1 a'_2 a'_3 a'_4$. By Lemma 9(iii), $N_{A'_S}(s_1)$ consists of two adjacent vertices of A'_S, namely a'_1 and without loss of generality a'_2. Now, Lemma 6 applied to the matching $\{s_1 a'_2, s_2 a_2\}$ implies that $G[\{a_1, a'_1, a'_3, a'_4\}]$ is randomly matchable. However, $a'_1 a'_3 \notin E(G)$ and a'_4 is adjacent to all three vertices a_1, a'_1 and a'_3, thus, $G[\{a_1, a'_1, a'_3, a'_4\}]$ is neither a C_4 nor a clique, a contradiction. Therefore, $G[A'_S + a_1 - a'_1]$ is a clique and consequently A'_S is a K_{2p} for some $p \geq 1$. This implies that $G[A'_S + a_1 - a']$ is a K_{2p} for every $a' \in N_{A'_S}(s_1)$, i.e. a_1 is complete to $A'_S - a'$. Moreover, a_1 is adjacent to a' since $N_{A_S}(s_1)$ is a clique. We conclude that a_1 is complete to A'_S, i.e. that A_1 is a clique.
  Recall that S is strongly independent. The unique vertex of $I \setminus S$ is some $a' \in A'_S \subseteq A_1$. By Lemma 9(iv), $N_{A_1}(s_1) \cap N_{A_1}(s_2) = \emptyset$. In particular, $a_1 s_2 \notin E(G)$. It remains to determine the neighborhoods of a_2 and s_2. We proceed by considering two disjoint cases regarding the neighborhood of s_2.
  - $N_{A'_S}(s_2) \neq \emptyset$: In this case, we will show that G has all the properties of $\mathcal{G}_{22}$. Properties (ii) and (iii) are trivial. Since the third vertex of I is some $a' \in A'_S$, Property (iv) holds, too. It suffices to show that (i) will hold, namely that A_S is a clique. By Lemma 9(iv), this implies Property (v).

Suppose that a_2 is not complete to A_1, and let a be an arbitrary vertex of A_1 that is not adjacent to a_2. Then $I' = \{a, a_2, b\}$, where b is the single vertex of the component B_S, is an independent set, and $as_2 \notin E(G)$ since $N_{A_S}(s_2)$ is a clique by Lemma 9(ii). Since $N_{A'_S}(s_2) \neq \emptyset$, $G \setminus I'$ is connected, a contradiction. Therefore, a_2 is complete to A_1, concluding that A_S is a clique.

– $N_{A'_S}(s_2) = \emptyset$: We first assume that $s_1a_2 \notin E(G)$. In this case, we claim that for all $a' \in A'_S$ such that $s_1a' \notin E(G)$, a_2 is adjacent to a'. Indeed, if $a_2a' \notin E(G)$ for such a vertex $a' \in A'_S$ then $S' = \{s_1, a_2\}$ is a strongly independent 2-cut (contained by the independent set $\{s_1, a_2, a'\}$) with $|B_{S'}| \geq 3$, a contradiction to the assumption of this proposition. So, assume in what follows that a_2 is adjacent to every vertex in $A'_S \setminus N_{A'_S}(s_1)$. Now, we will show that G has all the properties of $\mathcal{G}_{23}$ using the independent 2-cut $S' = \{s_1, a_2\}$. Properties (i), (ii), and (iii) are trivial since in this case $A_{S'} = \{s_2, b\}$ and $B_{S'} = A_1$. We now show Property (iv). Since a_2 is adjacent to every vertex in $A'_S \setminus N_{A'_S}(s_1)$ and $s_1a_1 \in E(G)$, we have that $N_{B_{S'}}(a_2) \cup N_{B_{S'}}(s_1) = B_{S'}$. Moreover, $N_{B_{S'}}(s_1) \neq \emptyset$ since $s_1a_1 \in E(G)$. Finally, since $\{s_1, s_2\}$ is a strongly independent 2-cut, there is a vertex $a' \in A'_S \subseteq A_1$ which is not adjacent to s_1 and consequently $a_2a' \in E(G)$ implying that $N_{B_{S'}}(a_2) \neq \emptyset$ and $N_{B_{S'}}(s_1) \neq B_{S'}$.
Now assume that $s_1a_2 \in E(G)$. In this case, we set $V_1 = A_1$ and show that G has all the properties of $\mathcal{G}_{21}$. Property (i) holds since A_1 is a clique, and (ii) holds since $V(G) \setminus A_1$ is the cycle $s_1a_2s_2b$. Property (v) holds since b and s_2 do not have neighbors in A_1. We now show that (iii) holds. $N_{A_1}(a_2) \subseteq N_{A_1}(s_1)$ since otherwise a_2, s_1, s_2 and a fourth vertex that is adjacent to a_2 and non-adjacent to s_1 form a claw. Furthermore, $N_{A_1}(s_1) \subseteq N_{A_1}(a_2)$ since otherwise s_1, a_2, b and a fourth vertex adjacent to s_1 and non-adjacent to a_2 form a claw. We now proceed to Property (iv). Since $N_{A'_S}(s_1) \neq \emptyset$ and $s_1a_1 \in E(G)$, we have $|N_{A_1}(s_1)| \geq 2$. Moreover, $N_{A_1}(s_1) \neq A_1$ since otherwise $\alpha(G) = 2$. This concludes the proof. □

Let us summarize the results of this section in the following:

Proposition 23. *If G is an equimatchable claw-free odd graph with $\alpha(G) \geq 3$ and $\kappa(G) = 2$, then $G \in \mathcal{G}_{21} \cup \mathcal{G}_{22} \cup \mathcal{G}_{23} + C_7$.*

Proof. Let S be a 2-cut of G. By Lemma 15, $G \setminus S$ consists of an even component A_S and an odd component B_S. Proposition 18 proves that if for some 2-cut S we have that A_S is a C_4, then $G \in \mathcal{G}_{21}$. In what follows we assume that for every 2-cut S of G, A_S is not a C_4.

By Corollary 16, G contains a strongly independent 2-cut. We consider the set $\mathcal{S} \neq \emptyset$ of all the strongly independent (minimal) 2-cuts, and consider the following disjoint and complementing subcases:

- There exists some $S' \in \mathcal{S}$ such that $|B_{S'}| > 1$ and $B_{S'}$ is not a P_3. In this case by Proposition 20, $G \in \mathcal{G}_{22}$.
- There exists some $S' \in \mathcal{S}$ such that $B_{S'}$ is a P_3. In this case, by Proposition 21, G is either a C_7 or a graph of $\mathcal{G}_{23}$.
- $|B_{S'}| = 1$ for every $S' \in \mathcal{S}$. In this case, by Proposition 22, we have that $G \in \mathcal{G}_{21} \cup \mathcal{G}_{22} \cup \mathcal{G}_{23}$. □

3.3. *Equimatchable claw-free odd graphs with $\alpha(G) \geq 3$ and $\kappa(G) = 1$*

Let us finally consider equimatchable claw-free odd graphs with independence number at least 3 and connectivity 1. We will show that these graphs fall into the following family.

Definition 3. Graph $G \in \mathcal{G}_1$ if it has cut vertex v where $G - v$ consists of two connected components G_1, G_2 such that for $i \in \{1, 2\}$

(i) Component G_i is either an even clique or a C_4.
(ii) If G_i is a C_4, then $N_{G_i}(v)$ consists of two adjacent vertices of G_i.
(iii) If both G_1 and G_2 are cliques, then v has at least one non-neighbor in each one of G_1 and G_2.

We note that $\mathcal{G}_1 = \{G_{11}\} \cup \mathcal{G}_{12} \cup \mathcal{G}_{13}$ where

$$\mathcal{G}_{12} = \left\{G_{12}(x, 2p - x, 1, 2p' - x', x') |\ 1 \leq x \leq 2p - 1, 1 \leq x' \leq 2p' - 1\right\},$$
$$\mathcal{G}_{13} = \{G_{13}(x, 2p - x, 1, 1, 1, 1, 1) |\ 0 \leq x \leq 2p - 1\}$$

where G_{11}, G_{12}, G_{13} are the graphs depicted in Fig. 2a, b and c, respectively.

Proposition 24. *If $G \in \mathcal{G}_1$, then G is a connected equimatchable claw-free odd graph with $\alpha(G) \geq 3$ and $\kappa(G) = 1$.*

Proof. All the other properties being easily verifiable, we will only show that G is equimatchable using Lemma 7. Note that $V(G_i) \setminus N(v)$ is a non-empty clique where v is a cut vertex of G. Therefore, every independent set I with three vertices containing v has exactly one vertex from every G_i. In this case, $G \setminus I$ has two odd components. An independent set I' with three vertices that does not contain v must contain two non-adjacent vertices of a C_4 and one vertex from the other component. Then one vertex of that C_4 is isolated in $G \setminus I'$. Let v' be the unique vertex of I' in the other component G_i. If v is a cut vertex of G (which happens when G_i is an even clique and $N_{G_i}(v) = \{v'\}$), then $G_i - v'$ constitutes a second odd connected component of $G \setminus I'$; otherwise, $G \setminus I'$ consists of two connected components and they are both odd. □

Proposition 25. *If G is an equimatchable claw-free odd graph with $\alpha(G) \geq 3$ and $\kappa(G) = 1$, then $G \in \mathcal{G}_1$.*

Proof. By Lemma 9(i), every cut vertex of G separates it into two connected components G_1 and G_2. From parity considerations, G_1 and G_2 are either both even or both odd. We consider two complementing cases:

- **Graph G has a cut vertex v such that G_1 and G_2 are even.** Let u be a vertex of G_1 adjacent to v. Considering the matching M consisting of the single edge uv and applying Lemma 6(iii), we conclude that G_2 is randomly matchable, i.e., either an even clique or a C_4 by Proposition 4. By symmetry, the same holds for G_1; thus, (i) in Definition 3 holds. Assume that G_i is a C_4 for some $i \in \{1, 2\}$. Then, by Lemma 9(iii), v is adjacent to exactly two adjacent vertices of G_i; thus, (ii) in Definition 3 holds. Finally, since $\alpha(G) \geq 3$, (iii) in Definition 3 also holds.
- **Every cut vertex v of G separates it into two odd components.** We will conclude the proof by showing that this case is not possible. No two cut vertices of G are adjacent, since otherwise one of them disconnects G into two even components. Let v be a cut vertex, G_1 and G_2 be the connected components of $G - v$, and u_1 be a neighbor of v in G_1. Applying Lemma 6 (iii) to the matching consisting of the single edge u_1v, we conclude that $G_1 - u_1$ is randomly matchable. Then, either $G_1 - u_1$ is connected, or by Lemma 9(i), $G_1 - u_1$ has exactly two connected components. Moreover, since u_1 is not a cut vertex of G, v has a neighbor in each of these components. If there are two such components, the neighbors of v in these components do not form a clique, contradicting Lemma 9(ii). Therefore, $G_1 - u_1$ is connected, and by Proposition 4 we conclude that it is either a C_4 or an even clique.
  Suppose that $G_1 - u_1$ is a C_4, say $w_1w_2w_3w_4$. By Lemma 9(iii), $N_{G_1-u_1}(v)$ consists of two adjacent vertices, say w_1, w_2. Consider the matching $M = \{vw_1, w_2w_3\}$. $V(M)$ disconnects $\{u_1, w_4\}$ from G. If u_1 and w_4 are non-adjacent, they contradict Lemma 6(ii). Therefore, u_1 is adjacent to w_4. Now the matching $M = \{vw_2, u_1w_4\}$ disconnects the vertices w_1 and w_3 from G and leaves two odd components, contradicting Lemma 6(ii). Hence, we conclude that $G_1 - u_1$ cannot be a C_4 and therefore has to be an even clique.
  We now show that u_1 is complete to $G_1 - u_1$. Suppose that there exists a vertex z of $G_1 - u_1$ that is non-adjacent to u_1. Then z is non-adjacent to v since otherwise v has two non-adjacent vertices, namely u_1 and z, in its neighborhood in G_1, a contradiction by Lemma 9(ii). Let z' be a vertex of $G_1 - u_1$ that is adjacent to v. Recall that such a vertex exists since u_1 is not a cut vertex of G, and clearly, $z \neq z'$. Now consider the matching consisting of the edge vz' and a perfect matching of the even clique $G_1 \setminus \{u_1, z, z'\}$. This matching leaves u_1 and z as two odd components, a contradiction by Lemma 6(ii). Therefore, G_1 is an odd clique, and v is adjacent to at least two vertices (namely, u_1 and z') of G_1. By symmetry, the same holds for G_2.
  Since $\alpha(G) \geq 3$, v is not adjacent to some vertex w_1 of G_1 and some vertex w_2 of G_2. Then $S = \{v, w_1, w_2\}$ is an independent set of G and $G \setminus S$ consists of two even components, contradicting Lemma 7. $\square$

4. Summary and recognition algorithm

In this section we summarize our results in Theorem 26 and use it to develop an efficient recognition algorithm.

Theorem 26. *A graph G is a connected claw-free equimatchable graph if and only if one of the following holds:*

(i) *G is a C_4.*
(ii) *G is a K_{2p} for some $p \geq 1$.*
(iii) *G is odd and $\alpha(G) \leq 2$.*
(iv) *$G \in \mathcal{G}_1$.*
(v) *$G \in \mathcal{G}_{21} \cup \mathcal{G}_{22} \cup \mathcal{G}_{23} + C_7$.*
(vi) *$G \in \mathcal{G}_3$.*

Proof. One direction follows from Proposition 4, Lemma 5 and Propositions 24, 17 and 10 in the order of the items from (i) to (vi). We proceed with the other direction. Let G be an equimatchable claw-free graph. If G is even, then by Proposition 4, it is either a C_4 or an even clique. It remains to show that if G is odd and $\alpha(G) \geq 3$, then G is either a C_7 or in one of the families $\mathcal{G}_1, \mathcal{G}_{21}, \mathcal{G}_{22}, \mathcal{G}_{23}, \mathcal{G}_3$. If $\kappa(G) = 1$, then $G \in \mathcal{G}_1$ by Proposition 25. If $\kappa(G) = 2$, then $G \in \mathcal{G}_{21} \cup \mathcal{G}_{22} \cup \mathcal{G}_{23} + C_7$ by Proposition 23. If $\kappa(G) = 3$, then $G \in \mathcal{G}_3$ by Proposition 13. $\square$

The recognition problem of claw-free equimatchable graphs is clearly polynomial since each one of the properties can be tested in polynomial time. Equimatchable graphs can be recognized in time $\mathcal{O}(m\bar{m})$ (see [2]), where m (respectively $\bar{m}$) is the number of edges (respectively non-edges) of the graph. Claw-free graphs can be recognized in $\mathcal{O}\left(m^{\frac{\omega+1}{2}}\right)$ time, where ω is the exponent of the matrix multiplication complexity (see [14]). The currently best exponent for matrix multiplication is $\omega \approx 2.37286$ (see [15]), yielding an overall complexity of $\mathcal{O}\left(m(\bar{m} + m^{0.687})\right)$.

We now show that our characterization yields a more efficient recognition algorithm.

Algorithm 1 Claw-free equimatchable graph recognition

Require: A graph G.

1: **if** G is even **then**
2: **return** (G is a clique or G is a C_4).
3: **if** $\bar{G}$ is triangle free **then return true**.
4: **if** G is a C_7 or G is a G_{11} **then return true**.

5: Compute the unique twin-free graph H and multiplicities $n_1, \ldots, n_k$ such that $G = H(n_1, \ldots, n_k)$.
6: **if** H is isomorphic to neither one of $G_{12}, G_{13}, G_{21}, G_{22}, G_{23}, G_3$ nor to a relevant subgraph of it **then**
7: **return false**
8: **else**
9: let H be isomorphic to G_x for some $x \in \{12, 13, 21, 22, 23, 3\}$ or to a relevant subgraph of it.
10: **return true** if and only if $(n_1, \ldots, n_k)$ matches the multiplicity pattern in the definition of $\mathcal{G}_x$.

Corollary 27. *Algorithm 1 can recognize equimatchable claw-free graphs in time* $\mathcal{O}\left(m^{1.407}\right)$.

Proof. The correctness of Algorithm 1 is a direct consequence of Theorem 26. As for its time complexity, Step 2 can be clearly performed in linear time. Step 3 can be performed in time $\mathcal{O}\left(m^{\frac{2\omega}{\omega+1}}\right) = \mathcal{O}\left(m^{1.407}\right)$ (see [1]).

For every graph G there is a unique twin-free graph H and a unique vector $(n_1, \ldots, n_k)$ of vertex multiplicities such that $G = H(n_1, \ldots, n_k)$. The graph H and the vector $(n_1, \ldots, n_k)$ can be computed from G in linear time using partition refinement, i.e. starting from the trivial partition consisting of one set, and iteratively refining this partition using the closed neighborhoods of the vertices (see [9]). Each set of the resulting partition constitutes a set of twins. Therefore, Step 5 can be performed in linear time.

We now note that for some values of $x \in \{12, 13, 21, 22, 23, 3\}$, at most one entry of the multiplicity vector is allowed to be zero. In this case H is not isomorphic to G_x but to an induced subgraph of it with one specific vertex removed. We refer to these graphs as *relevant subgraphs* in the algorithm.

As for Step 6, it takes a constant time to decide whether an isomorphism exists: if H has more than 9 vertices, it is isomorphic to neither one of $G_{11}, G_{12}, G_{13}, G_{21}, G_{22}, G_{23}, G_3$ nor to a subgraph of them; otherwise, H has to be compared to each one of these graphs and their relevant subgraphs, where each comparison takes constant time. Finally, Step 10 can be performed in constant time.

We conclude that the running time of Algorithm 1 is dominated by the running time of Step 3, i.e. $\mathcal{O}\left(m^{1.407}\right)$. □

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