Last update: May 5, 2023

The `Libre-Office` ILP solver

An company has the opportunity of investing in $n \in \mathbb{N}_+$ different projects, if a project j ($j \in \{1, 2, \ldots, n\}$) is selected it generates a profit $p_j \in \mathbb{R}_+$ and it has a costs $w_j \in \mathbb{R}_+$. The company has a total budget $c \in \mathbb{R}_+$ and its goal is to determine a subset of projects of maximum total profit that respects its budget.

Questions

1. Write a integer linear programming (ILP) model to determine a subset of projects of maximum total profit which can be selected by the company, i.e,. a subset of projects of total cost no larger than the budget. Identify the decisions that must be taken and the corresponding decision variables. Identify and comment the objective function of the problem and the constraints.

2. Consider now the instance in which there are five projects ($n = 5$) with the following profits and costs:

j	1	2	3	4	5
p_j	8	6	14	6	2
w_j	3	3	9	5	2

   The budget of the company is $c = 11$.

   Write the ILP model for this instance and find an optimal solution and the optimal solution value using the ILP solver of `Libre-Office`.

3. Consider now these additional constraints to be added to the ILP model of the previous instance:

   - at most two out of the five projects can be selected;
   - if project 2 is selected, then project 4 must also be selected;
   - if project 1 is selected, then project 3 cannot be selected.

   Find an optimal solution and the optimal solution value using the ILP solver of `Libre-Office`.

Solution

1. We introduce the following n binary variables:

$$x_j = \begin{cases} 1 & \text{if project } j \text{ is selected} \\ 0 & \text{otherwise} \end{cases} \qquad \forall\, j \in \{1,2,\dots,n\},$$

Using these variables, an ILP model for the problem reads as follows:

$$\max \sum_{j=1}^{n} p_j\, x_j \tag{1a}$$

$$\text{subject to} \quad \sum_{j=1}^{n} w_j\, x_j \le c, \tag{1b}$$

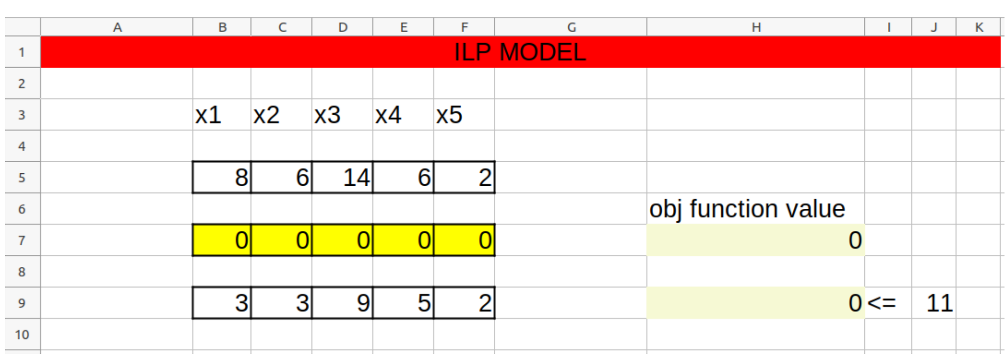
$$x_j \in \{0,1\}, \qquad j \in \{1,2,\dots,n\}. \tag{1c}$$

The objective function (1a) maximizes the total profit of the selected projects. Constraint (1b) ensures that the budget is respected. Finally constraints (1c) define the variables of the ILP model.

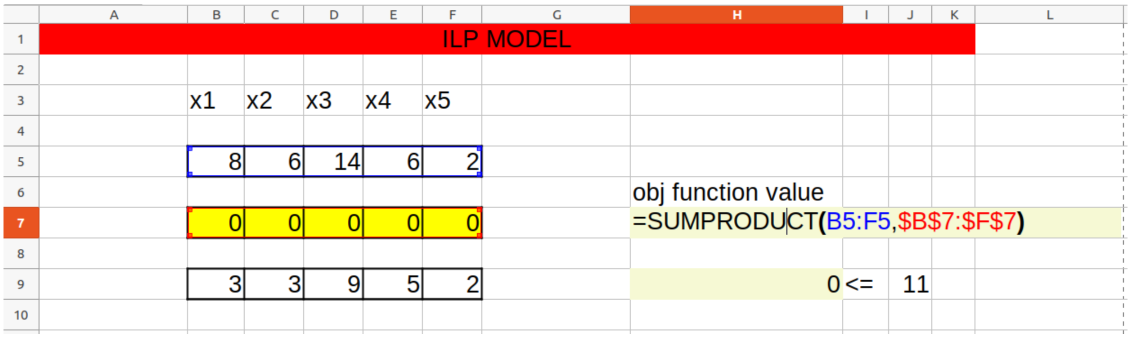
2. The ILP model (1) for the specific instance reads as follows:

$$\begin{array}{lllllll} \max & 8\ x_1 & +6\ x_2 & +14\ x_3 & +6\ x_4 & +2\ x_5 & & \\ & 3\ x_1 & +3\ x_2 & +9\ x_3 & +5\ x_4 & +2\ x_5 & \le & 11 \\ & x_1, & x_2, & x_3, & x_4, & x_5 & \in & \{0,1\} \end{array}$$

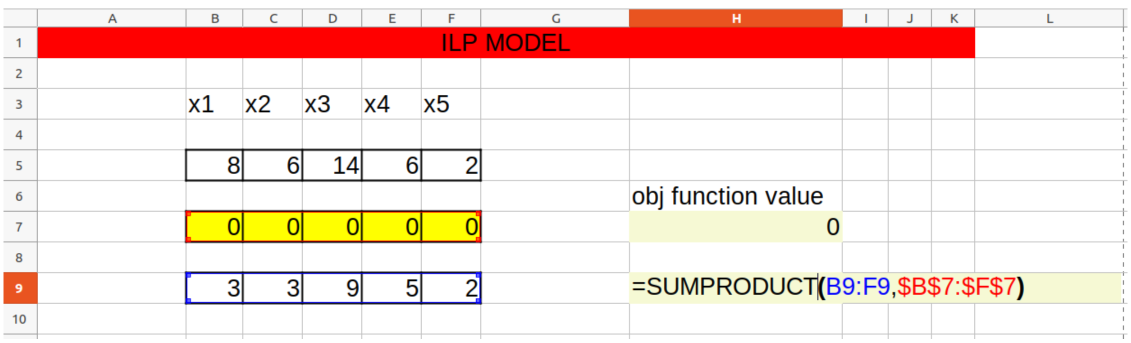
We now show how to compute an optimal solution of the ILP model for the previous instance using the ILP solver of `Libre-Office`. The spreadsheet of the model is presented by the following figure, the yellow cells represent the values of the five variables of the ILP model.



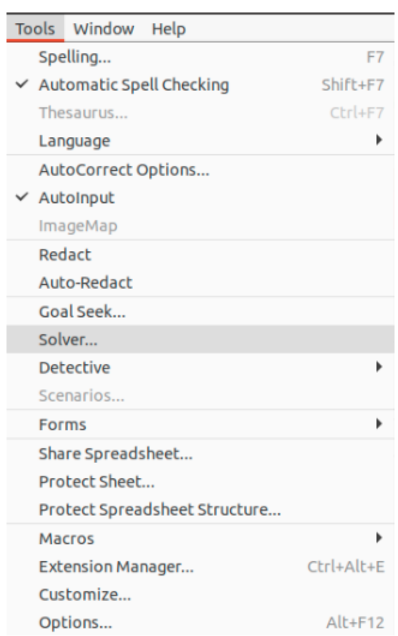
Using the SUMPRODUCT function, we can compute the objective function value of the problem in function of the values given to the variables and the objective-function vector as follows:



Using again the SUMPRODUCT function, we can also compute the left-hand-side of the constraint in function of the values given to the variables and the constraint vector as follows:

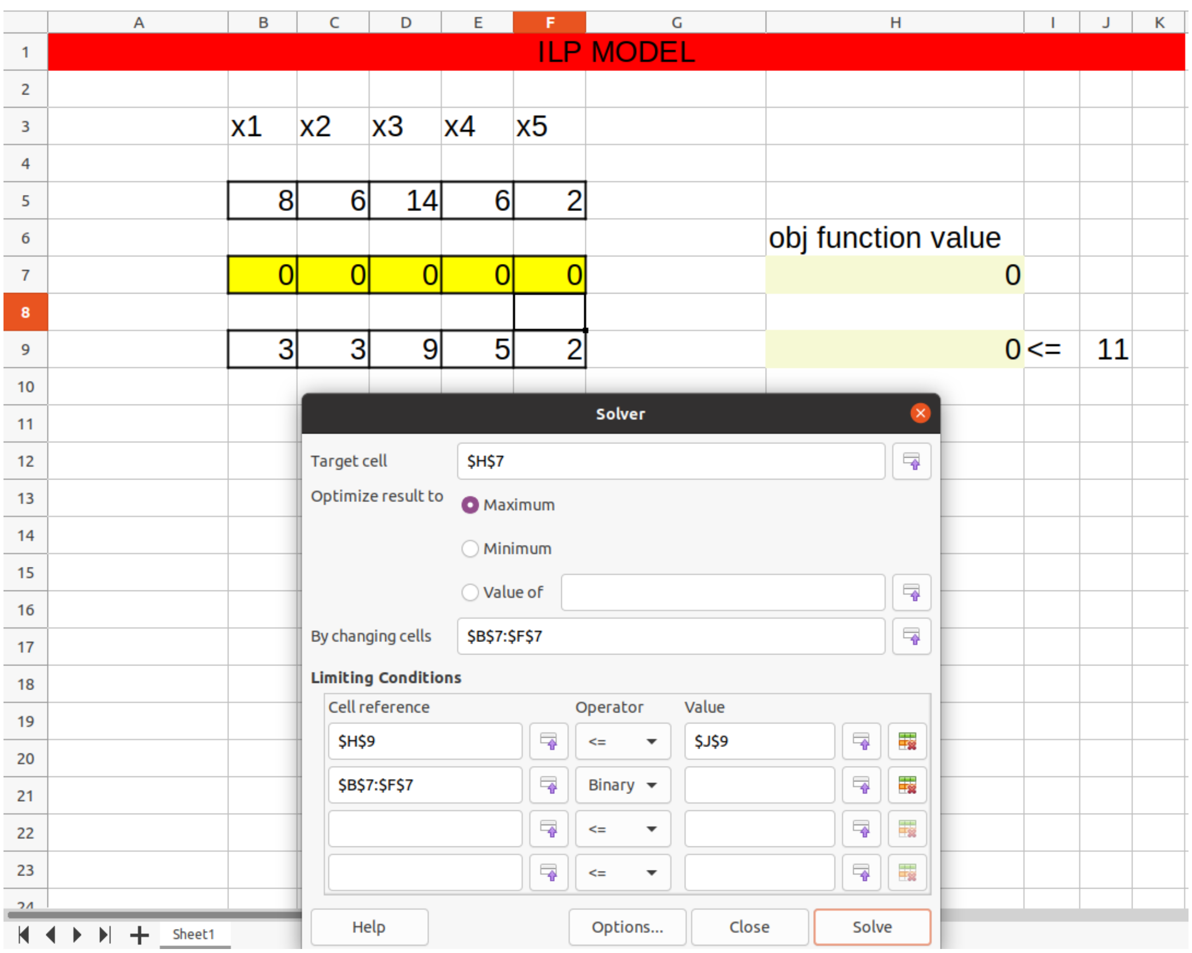


The ILP solver of Libre-Office is then called as follows:

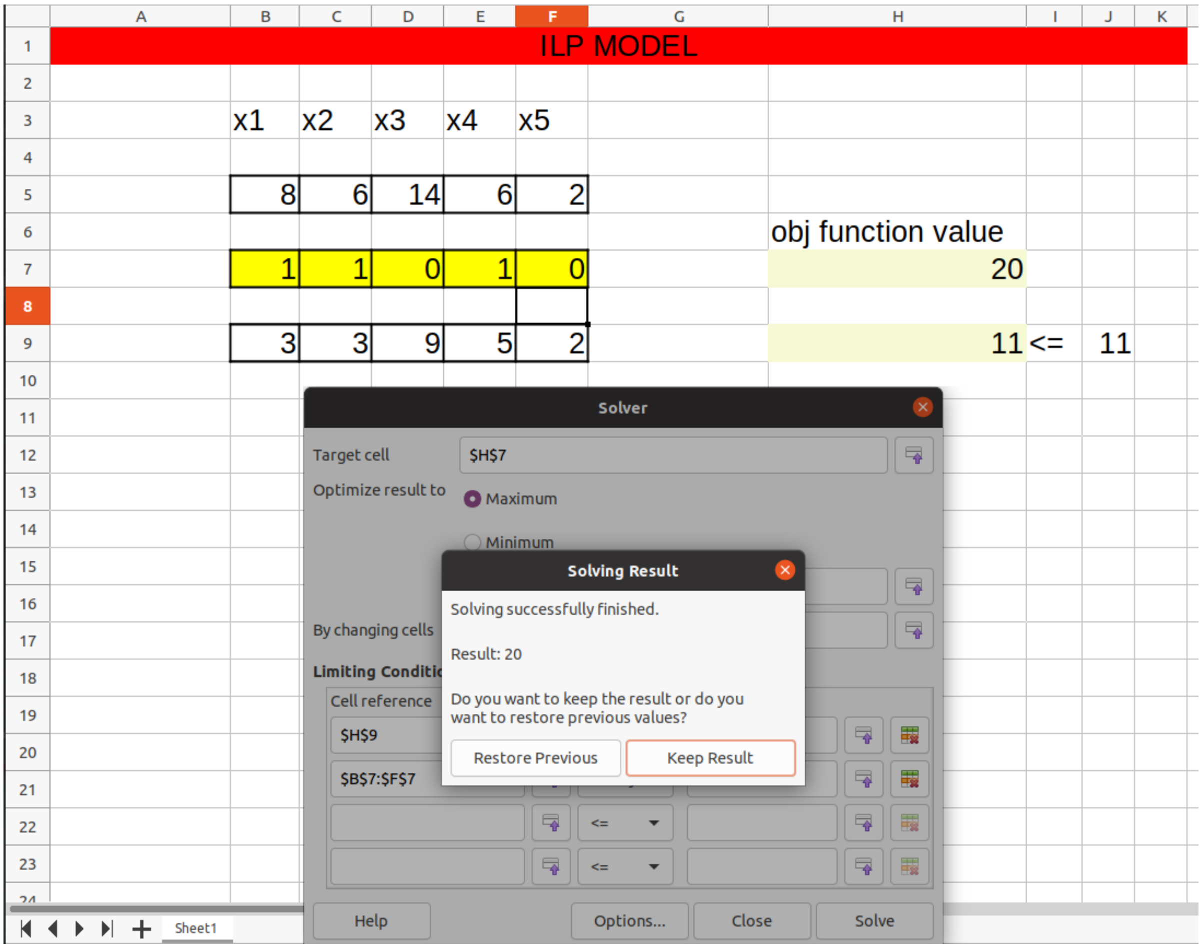


The information of the model are given to the solver in the following manner:

- the cell of the objective function value is put into the "Target cell"
- the sense of the objective function is determined in "Optimize results to" ("Maximum" in this case)
- the cells representing the values of the variables of the problem are put into "By changing cells"
- the information related to the constraints of the model are given to the solver in the part "Limiting conditions". Each constraint is inserted by giving two cells, i.e., the left-hand-side of the constraint and the right-and-side of the constraint. Then the type of the constraint is imposed (in this case "≤").
- finally, since all the variables are binary, the cells of the values of the variables are set as "Binary"



Pressing the button"Solve", the solver computes an optimal solution and the optimal solution value of the ILP model as shown in the following figure:



An optimal solution is:

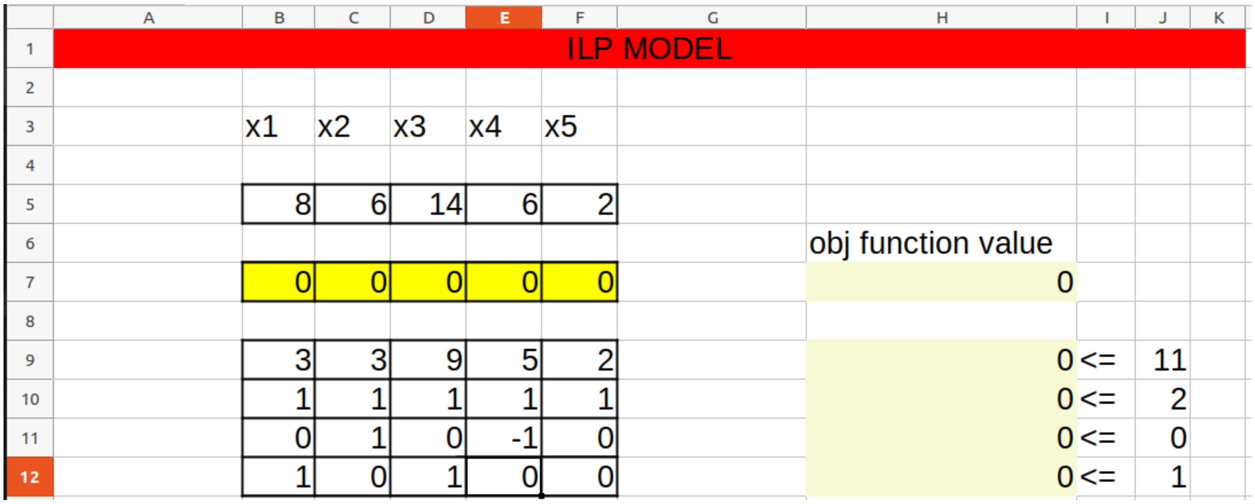
$$x_1^* = 1, \;\; x_2^* = 1, \;\; x_3^* = 0, \;\; x_4^* = 1, \;\; x_5^* = 0.$$

The optimal solution value is $z(\text{ ILP }) = 20$

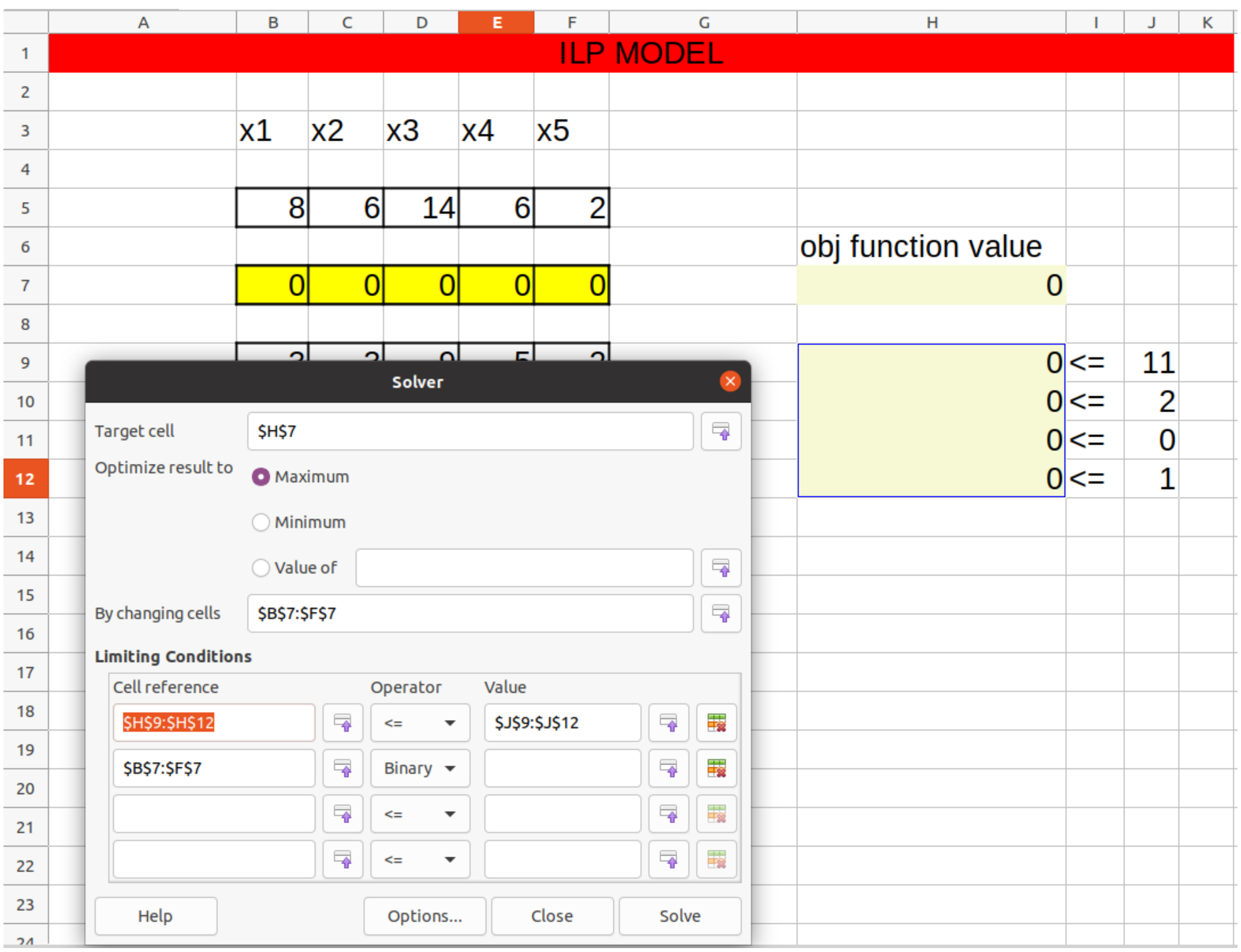
3. The ILP model with the additional constraints reads as follows:

$$\begin{array}{lrrrrrcc}
\max & 8\;x_1 & +6\;x_2 & +14\;x_3 & +6\;x_4 & +2\;x_5 & & \\
 & 3\;x_1 & +3\;x_2 & +9\;x_3 & +5\;x_4 & +2\;x_5 & \leq & 11 \\
 & x_1 & +\;x_2 & +\;x_3 & +\;x_4 & +\;x_5 & \leq & 2 \\
 & & x_2 & & -\;x_4 & & \leq & 0 \\
 & x_1 & & +\;x_3 & & & \leq & 1 \\
 & x_1, & x_2, & x_3, & x_4, & x_5 & \in & \{0,1\}
\end{array}$$

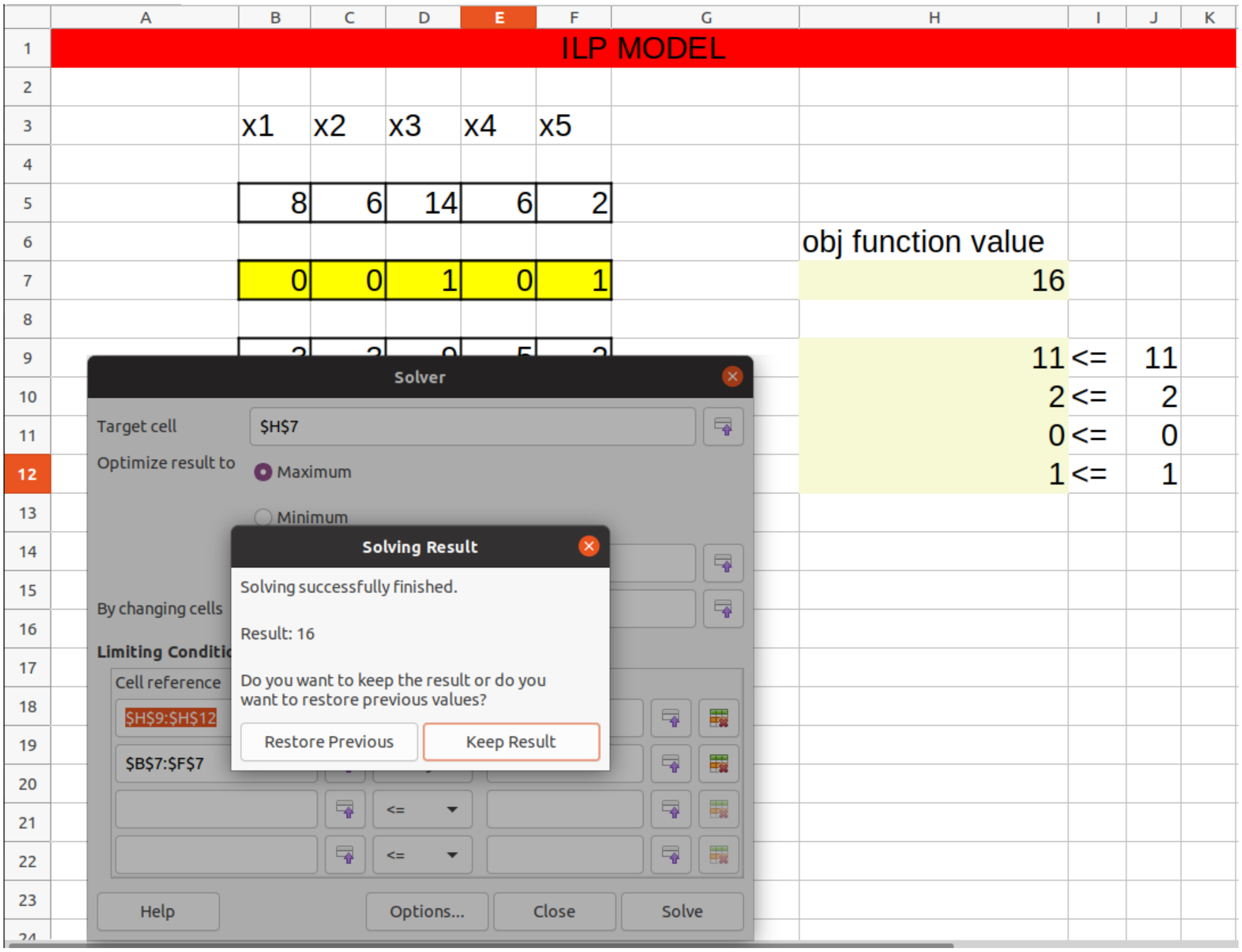
The spreadsheet of the model is presented by the following figure:



The constraints can be inserted altogether as follows:



And this is the new optimal solution and the optimal solution value of the ILP model with the additional constraints:



A new optimal solution is:

$$x_1^* = 0, \;\; x_2^* = 0, \;\; x_3^* = 1, \;\; x_4^* = 0, \;\; x_5^* = 1$$

The new optimal solution value is $z(\text{ ILP }) = 16$.