1

Dominating Sequences in Graphs

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Outline

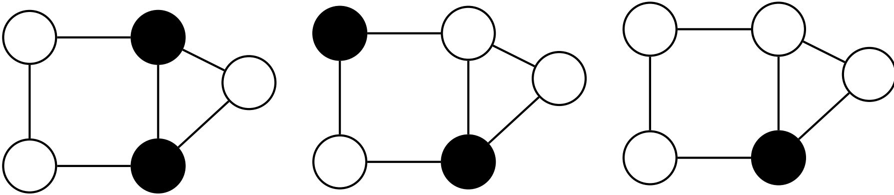
Preliminaries

- Let $G = (V(G), E(G))$ be a simple graph with vertex set $V(G)$ and edge set $E(G)$
- The *(open) neighborhood* of a vertex $v \in V(G)$, denoted by $N(v)$, is the set of vertices adjacent to v
- $N[v] = N(v) \cup \{v\}$ = The *closed neighborhood* of a vertex $v \in V(G)$
- The degree of a vertex u is $deg(u) = |N(v)|$ and minimum degree is denoted by $\delta(G)$

Dominating Set

A subset $S \subseteq V(G)$ is called a *dominating set* of G if every vertex in $V(G) \setminus S$ has at least one neighbor in S

$\gamma(G)$ = Domination number = Size of a minimum dominating set



Dominating Sequences

A sequence $S = (v_1, \ldots, v_k)$ of distinct vertices in a graph G is called a *legal dominating sequence* if every vertex in the sequence dominates at least one vertex not dominated by those vertices that precede it, and at the end all vertices of G are dominated; that is, if

$N[v_i] \setminus \bigcup_{j=1}^{i-1} N[v_j] \neq \emptyset$ holds for every $i \in \{2, \ldots, k\}$.

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$N[v_i] \setminus \bigcup_{j=1}^{i-1} N[v_j] \neq \emptyset$ holds for every $i \in \{2, \ldots, k\}$.

Clearly, the length k of a legal dominating sequence $S = (v_1, \ldots, v_k)$ is bounded from below by the domination number $\gamma(G)$.

Grundy Domination Number

Bresar et.al. introduced the notion of *Grundy domination number* $\gamma_{gr}(G)$ as the maximum length of a legal dominating sequence.

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The lower bound $\gamma_{gr}(G) \geq \gamma(G)$ is obvious.

Proposition

For an arbitrary graph G, $\gamma_{gr}(G) \leq |V(G)| - \delta(G)$

Grundy Domination Number

Proof.

Let $S = (s_1, ..., s_k)$ be a Grundy dominating sequence of G and let u be a vertex footprinted in the last step. Since u is not dominated before the last step, we have:

$N[u] \cap \{s_1, ..., s_{k-1}\} = \emptyset$, and so

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$|\{s_1, ..., s_{k-1}\}| = k - 1 \leq |V(G)| - (deg(u) + 1).$

Thus, $\gamma_{gr}(G) = k \leq |V(G)| - \delta(G)$ $\square$

k-Uniform Graphs

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A graph G is said to be *twin-free* if no two of its distinct nodes are twins.

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A graph G is said to be *twin-free* if no two of its distinct nodes are twins.

Creating a twin of a vertex changes neither the domination number nor the Grundy domination number

Therefore, the question of characterizing k-uniform graphs is interesting only for twin-free graphs

Characterization of k-Uniform Graphs for $k \leq 3$

We will review the following work:

B.Bresar, T.Gologranc, M.Milanic, D.F.Rall, R.Rizzi, "Dominating sequences in graphs", *Discrete Mathematics*, vol.336, pp.22-36, 2014.

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Lemma

Let G be a twin-free k-uniform graph and let $x, y \in V(G)$. If $N[x] \subseteq N[y]$, then $x = y$.

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Suppose that $N[x] \subseteq N[y]$. If $N[x] = N[y]$, then we must have $x = y$ since G is twin-free. Hence, we may assume that there exists a vertex $z \in N(y) \setminus N(x)$.

Since G is k-uniform for some k, the sequence $S = (x, y)$ can be extended to a legal dominating sequence S' of G of length k, say $S' = (x, y, v_3, \ldots, v_k)$.

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Since G is k-uniform for some k, the sequence $S = (x, y)$ can be extended to a legal dominating sequence S' of G of length k, say $S' = (x, y, v_3, \ldots, v_k)$. But then, $(y, v_3, \ldots, v_k)$ is a legal dominating sequence of G of length $k - 1$, contradicting the fact that G is k-uniform. $\square$

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Let $G = (V, E)$ be any graph and $S \subseteq V$. Then the induced subgraph $G[S]$ is the graph whose vertex set is S and whose edge set consists of all of the edges in E that have both endpoints in S.

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P_n is a path on n vertices

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An *independent set* is a set of vertices in a graph, no two of which are adjacent.
$\alpha(G)$ denotes the maximum size of an independent set.

Proof.

($\Leftarrow$) Easy.

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If G is 2-uniform, then $|V(G)| \geq 2$ and G is P_4-free.

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If G is 2-uniform, then $|V(G)| \geq 2$ and G is P_4-free.
Suppose that G is disconnected ... $\overline{G}$ is $K_{r,s}$ $\square$

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Claim: Every two distinct vertices of G have exactly one common non-neighbor.

Proof of Claim: ...

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Therefore, every two distinct vertices have a unique common neighbor in $\overline{G}$. By a theorem of Erdös, $\overline{G}$ is isomorphic to a *friendship graph*, that is, a graph obtained from the disjoint union of n copies of K_2 by adding to it a universal vertex.

By adding twins to the friendship graph, we see that the theorem is true.

Characterization of k-Uniform Graphs for $k \geq 4$

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A.Erey, "Uniform length dominating sequence graphs", *Graphs and Combinatorics*, vol.36, pp.1819-1825, 2020.

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Lemma

Let G be a k-uniform graph and v be any vertex of G. Then,

- *the subgraph $G \setminus N[v]$ is $(k-1)$-uniform,*
- *if G has no true twins, then $G \setminus N[v]$ has no true twins.*

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  $(w_2, w_3, \ldots, w_k)$ is a legal dominating sequence of $G \setminus N[v]$....

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  $(w_1, w_2, w_3, \ldots, w_k, v)$ is also a legal dominating sequence of length $(k+1)$, contradiction.

$\square$

Characterization of k-Uniform Graphs for $k \geq 4$

Remark

Let G be a graph with connected components $G_1, \ldots, G_c$. Then G is k-uniform if and only if each G_i is k_i-uniform, where $k = k_1 + \ldots, k_c$ and $k_i \geq 1$.

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The *join* of two graphs G and H, denoted by $G \vee H$ has vertex set $V(G \vee H) = V(G) \cup V(H)$ and edge set $E(G \cup H) = E(G) \cup E(H) \cup \{uv \mid u \in V(G, v \in V(H))\}$.

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Let $\vee_t H$ denote the join of t copies of the graph H and tH denote disjoint union of t copies of H

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Let $G_1, \ldots, G_r$ be the 1-uniform connected components and $H_1, \ldots, H_t$ be the 2-uniform connected components of the subgraph $G \setminus N[v]$. ....

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Let us show that G is disconnected. Suppose to the contrary that G is connected. Let $A_i = N(v_i) \cap N(v)$ and $B_i = N(V(H_i)) \cap N(v)$ for each i.

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Since G is connected, A_i and B_i are nonempty for each i. $\square$

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- $B_i \cap B_j = \emptyset$ whenever $i \neq j$: Suppose that there exists a vertex $w \in N(v)$ such that w is adjacent to both u_1 and u_2. Then $(v, w, u_1^{'}, u_2^{'}, u_3, u_3^{'}, \ldots, v_1, \ldots, v_r)$ is a legal dominating sequence of length $r + 2t = k - 1$, contradiction.

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- $A_i \cap B_j = \emptyset$ for every i and j: Suppose that there exists a vertex $w \in N(v)$ such that w is adjacent to both v_1 and u_1. Then again, a legal dominating sequence of length $k - 1$ is found.

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- $A_i \cap B_j = \emptyset$ for every i and j: Suppose that there exists a vertex $w \in N(v)$ such that w is adjacent to both v_1 and u_1. Then again, a legal dominating sequence of length $k - 1$ is found.

Therefore, $A_1, \ldots, A_r, B_1, \ldots, B_t$ are mutually disjoint.

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We then show that G is disconnected and the result follows.