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Total Dominating Sequences in Graphs

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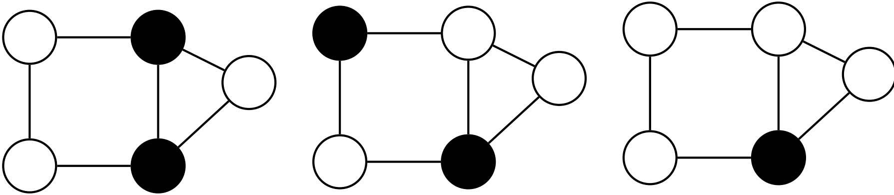
Total Dominating Set

If G has no isolated vertices, then a subset $S \subseteq V(G)$ is called a *total dominating set* of G if every vertex in $V(G)$ has at least one neighbor in S

$\gamma_t(G)$ = Total domination number = Size of a minimum total dominating set



Total Dominating Sequences

A sequence $S = (v_1, \ldots, v_k)$ of distinct vertices of G is a *legal (open neighborhood) sequence* if $N(v_i) \setminus \bigcup_{j=1}^{i-1} N(v_j) \neq \emptyset$ holds for every $i \in \{2, \ldots, k\}$.

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If, in addition, S is a total dominating set of G, then we call S a *total dominating sequence* of G.

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If, in addition, S is a total dominating set of G, then we call S a *total dominating sequence* of G.

Clearly, the length of a total dominating sequence in G is at least $\gamma_t(G)$, and any permutation of a minimum total dominating set of G forms a total dominating sequence attaining this lower bound.

Grundy Total Dominating Sequence

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$$\gamma_{gr}^t(G) = \gamma_{gr}(G) = k$$

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Observation

For every graph G, $\gamma_{gr}^t(G) \geq \max \gamma_{gr}^t(H)$, where the maximum is taken over all induced subgraphs H of G with no isolated vertex.

Background

- Numerous variants of Grundy total domination such as Grundy domination, Z-Grundy domination and L-Grundy domination exist in the literature (Bresar et.al.)

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- Relations between these types of Grundy domination as well as the relation between the Z-Grundy domination number and the zero forcing number of a graph have been investigated (Bresar et.al.)
- The concept of total domination was introduced in 1980 and have been studied extensively in the literature

Background

- The parameter $\gamma_{gr}^t(G)$ was first introduced by Bresar et.al. in 2016 who:
  - obtained bounds on $\gamma_{gr}^t(G)$ for trees, regular graphs, and graph products in terms of other graph variants

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  - showed that complete multipartite graphs are the only total 2-uniform graphs and there are no total 3-uniform graphs
  - provided an infinite family of total 4-uniform graphs

In this section, the following paper will be discussed:

B.Bresar, M.A.Henning, D.F.Rall, "Total dominating sequences in graphs", *Discrete Mathematics*, vol.339, no.6, pp.1665-1676, 2016.

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($\Rightarrow$) If x and y are adjacent, then $\{x, y\}$ is a total dominating set...
If x and y are nonadjacent, then $N(x) = N(y)$....
Hence, if A is a maximal independent set in G, then $N(x)$ of all vertices x from A coincide. $\square$

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Hence, G can be partitioned into maximal independent sets, each of which is adjacent to all other vertices not in that set. In other words, G is a complete multipartite graph.

Total 3-Uniform Graphs

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If G is a graph with $\gamma_t(G) = 3$, then $\gamma_{gr}^t(G) > 3$; that is, there is no graph G satisfying $\gamma_t(G) = \gamma_{gr}^t(G) = 3$.

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Let G be a graph with $\gamma_t(G) = 3$ and let $S = \{a, b, c\}$ be a minimum total dominating set of G. Since $G[S]$ contains no isolated vertex, either $G[S] = P_3$ or $G[S] = K_3$.

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Total 4-Uniform Graphs

Proposition

There are infinitely many connected graphs G with $\gamma_{gr}^t(G) = \gamma_t(G) = 4$.

Proof.

Infinite families $\{\mathcal{G}_m\}_{m \geq 3}$ of connected graphs with $\gamma_{gr}^t(G) = \gamma_t(G) = 4$ can be constructed as follows. Let $m \geq 3$ be an integer.

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For each i with $1 \leq i \leq m$, let X_i and Y_i be nonempty sets of vertices such that the sets $X_1, \ldots, X_m, Y_1, \ldots, Y_m$ are pairwise disjoint.

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Let $X = \cup_{i=1}^m X_i$ and $Y = \cup_{i=1}^m Y_i$. A bipartite graph G with $V(G) = X \cup Y$ is obtained by adding the edge xy if and only if $x \in X_i$ and $y \in Y_j$ for $i \neq j$.

$\square$

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..... Hence, $\gamma_t(G) = \gamma_{gr}^t(G) = 4$.

C_6 is the smallest graph in $\{\mathcal{G}_m\}_{m \geq 3}$.

In this section, the results in the following paper will be reviewed:

T.Dravec, M.Jakovac, T.Kos, T.Marc, "On graphs with equal total domination and Grundy total domination numbers", *Aequationes Mathematicae*, vol.96, pp.137-146, 2022.

Bipartite Total 4-Uniform Graphs

Theorem

Let G be a bipartite false twin-free graph. Then $\gamma_t(G) = \gamma_{gr}^t(G) = 4$ if and only if G is isomorphic to the graph $K_{n,n} - M$, $n \geq 2$, where M denotes an arbitrary perfect matching of $K_{n,n}$.

Chordal Total 4-Uniform Graphs

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There is no connected chordal graph G with $\gamma_t(G) = \gamma_{gr}^t(G) = 4$.

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Conjecture

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Bipartite Total 6-Uniform Graphs

For a bipartite graph G with bipartition sets A and B, the *bipartite complement* of G is the bipartite graph G^{bc} with the same bipartition sets, where $a \in A$ is adjacent to $b \in B$ in G^{bc} if and only if a is not adjacent to b in G.

Bipartite Total 6-Uniform Graphs

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Theorem

Let G be a connected bipartite false twin-free graph. Then $\gamma_t(G) = \gamma_{gr}^t(G) = 6$ if and only if the bipartite complement of G is the incidence graph of a projective plane.

The existence is closely related to the existence of finite projective planes, one of the oldest and still unsolved combinatorial questions.

So, the situation for higher values is much more complicated.

Bipartite Total 6-Uniform Graphs

Corollary

If G is a connected bipartite false twin-free graph such that $\gamma_t(G) = \gamma_{gr}^t(G) = 6$, then G is regular.

An example total 6-uniform regular bipartite graph is ......

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Corollary

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Corollary

Let G be a connected bipartite false twin-free graph. If $\gamma_t(G) = \gamma_{gr}^t(G) = 6$, then $|V(G)| = 2(k^2 + k + 1)$ for some integer $k > 1$.

Now we will focus on the following work:

S.Bahadır, D.Gözüpek, O.Doğan, "On graphs all of whose total dominating sequences have the same length", *Discrete Mathematics*, vol.344, no.9, 2021.

In this work, the authors extended the work of Bresar et.al. by :

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- presenting a total 4-uniform graph, disproving a conjecture by Dravec et.al.

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In this work, the authors extended the work of Bresar et.al. by :

- proving that there is no total k-uniform graph when k is odd
- presenting a total 4-uniform graph, disproving a conjecture by Dravec et.al.
- providing a connected total 8-uniform graph
- proving that every total k-uniform, connected and false twin-free graph is regular for every even k

A Reduction From Total k-Uniform Graphs to Total $(k-2)$-Uniform Graphs

Lemma

If $v_1 v_2$ is an edge of a total k-uniform graph G where $k \geq 2$, then $G \setminus (N[v_1] \cup N[v_2])$ has no isolated vertices

Proof.

..... $\square$

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Lemma

Let G be a total k-uniform graph with no isolated vertices where $k \geq 3$. If $uv \in E(G)$, then $G \setminus (N[u] \cup N[v])$ is a total $(k-2)$-uniform graph with no isolated vertex.

Proof.

..... □

A Reduction From Total k-Uniform Graphs to Total $(k-2)$-Uniform Graphs

Now for odd k, applying the previous lemma $(k-1)/2$ times beginning with a total k-uniform graph yields a total 1-uniform graph, which does not exist since $\gamma_t(G) \geq 2$ for every G with no isolated vertex:

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Now for odd k, applying the previous lemma $(k-1)/2$ times beginning with a total k-uniform graph yields a total 1-uniform graph, which does not exist since $\gamma_t(G) \geq 2$ for every G with no isolated vertex:

Theorem

There does not exist a total k-uniform graph where k is an odd positive integer

A Reduction From Total k-Uniform Graphs to Total $(k-2)$-Uniform Graphs

Bresar et.al. proved that:

Corollary

Let G be a graph. For any number ℓ such that $\gamma_t(G) \leq \ell \leq \gamma_{gr}^t(G)$, there is a total dominating sequence of G having length ℓ.

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Corollary

Let G be a graph. For any number ℓ such that $\gamma_t(G) \leq \ell \leq \gamma_{gr}^t(G)$, there is a total dominating sequence of G having length ℓ.

By combining these two results, we get:

Corollary

Every graph with no isolated vertex has a total dominating sequence of even length

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- Notice that removing one of the false twins or creating a false twin of a vertex changes neither the total domination number nor the Grundy total domination number

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- Two distinct vertices u and v of a graph G are called *false twins* if $N(u) = N(v)$
- A graph is *false twin-free* if it has no false twins.
- Notice that removing one of the false twins or creating a false twin of a vertex changes neither the total domination number nor the Grundy total domination number
- Therefore, the question of characterizing total k-uniform graphs is interesting only for false twin-free graphs

A Reduction From Total k-Uniform Graphs to Total $(k-2)$-Uniform Graphs

Let $\mathcal{G}_k$ be the family of total k-uniform and false twin-free graphs for every even k.

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Let $\mathcal{G}_k$ be the family of total k-uniform and false twin-free graphs for every even k.

By a previous theorem in the literature, the family of total 2-uniform graphs is the set of all complete multi-partite graphs

Therefore, $\mathcal{G}_2$ is the set of complete graphs with at least two vertices

A Reduction From Total k-Uniform Graphs to Total $(k-2)$-Uniform Graphs

Lemma

Let $k \geq 4$ be an even integer and $G \in \mathcal{G}_k$. Then for any edge $uv \in E(G)$ the graph $G \setminus (N[u] \cup N[v])$ is also false twin-free.

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Proof.

Let $H = G \setminus (N[u] \cup N[v])$. To the contrary, assume that H has some false twins, say w and w'......

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Let $H = G \setminus (N[u] \cup N[v])$. To the contrary, assume that H has some false twins, say w and w'......

By the previous lemma, we know that H is a total $(k-2)$-uniform graph. Thus, H has a total dominating sequence of length $k-2$ starting with w, say $(w = w_1, \ldots, w_{k-2})$.

Now consider the sequence $S = (w', w = w_1, \ldots, w_{k-2}, u, v)$. S is a legal sequence in G because .... $\square$

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Furthermore, S is also a total dominating set because ...

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Hence, S is a total dominating sequence of G of length $k+1$, contradiction. $\square$

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Proposition

Let $k \geq 4$ be an even integer. Let $G \in \mathcal{G}_k$ and uv be an arbitrary edge of G. Then $G \setminus (N[u] \cup N[v])$ is in $\mathcal{G}_{k-2}$.

Total k-Uniform Graphs with Small k

For a given graph G, its line graph $L(G)$ is a graph such that each vertex of $L(G)$ represents an edge of G and two vertices of $L(G)$ are adjacent if and only if their corresponding edges are incident.

Example line graph for $L(K_4)$ ...

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Proposition

The graph $L(K_6)$ is total 4-uniform.

Proof.

It is sufficient to prove that $\gamma_t(L(K_6)) \geq 4$ and $\gamma_{gr}^t(L(K_6)) \leq 4$.

We first show that $\gamma_t(L(K_6)) \geq 4$ .....

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We next show that $\gamma_{gr}^t(L(K_6)) \leq 4$. Assume to the contrary that $L(K_6)$ has a legal sequence $(v_1, v_2, v_3, v_4, v_5)$.... $\square$

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Clearly, $L(K_6)$ is connected and false twin-free but not a bipartite graph, hence disproving the conjecture by Gologranc et.al.

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We will now make use of graph products to obtain new total k-uniform graphs.

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We will now make use of graph products to obtain new total k-uniform graphs.

The *direct product* $G \times H$ of graphs G and H is a graph such that the vertex set of $G \times H$ is the Cartesian product $V(G) \times V(H)$ and vertices (g, h) and (g', h') are adjacent in $G \times H$ if and only if $gg' \in E(G)$ and $hh' \in E(H)$.

Total k-Uniform Graphs with Small k

We obtain the following, which enables us to create a connected total $2k$-uniform graph based on a connected non-bipartite total k-uniform graph:

Theorem

If the graph G is connected, non-bipartite and total k-uniform, then $G \times K_2$ is a connected total $2k$-uniform graph.

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Corollary

The graph $L(K_6) \times K_2$ is connected and total 8-uniform.

Regularity

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Let G be a bipartite, connected, false twin-free and total k-uniform graph for some even positive integer k. Then G is a regular graph.

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Let X and Y be the parts of G. We first prove that any two vertices in X sharing a common neighbor... $\square$

Regularity

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For every positive even integer k, a connected, false twin-free and total k-uniform graph is regular.

Total k-Uniform Chordal Graphs

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Lemma

If G is a connected graph in $\mathcal{G}_k$ where $k \geq 4$, then G has an induced C_5 or C_6.

Proof.

.... □

Total k-Uniform Chordal Graphs

Note that removing a false twin from a connected chordal graph affects neither connectedness nor chordality. Then, since any total k-uniform graph has a subgraph in $\mathcal{G}_k$, we obtain the following:

Theorem

For any $k \geq 4$, there does not exist a total k-uniform connected chordal graph

Total k-Uniform Chordal Graphs

This work then presents a complete characterization of total k-uniform chordal graphs:

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Let G be a graph without isolated vertices. G is a total k-uniform chordal graph if and only if G is disjoint union of complete multipartite graphs in which at most one part is of size greater than 1.

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The *girth* of a graph, denoted by $g(G)$, is the length of a shortest cycle (if any) in G.

Theorem

If G is a total k-uniform graph, then either G is disjoint union of $k/2$ stars or $g(G) \leq 6$.

Open Research Directions

For every integer l satisfying $\gamma_t(G) \leq l \leq \gamma_{gr}^t(G)$, there exists a total dominating sequence of length l in G (Bresar et.al.).

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One research direction is to solve the decision problem of determining whether a given graph is total k-uniform for some k.

In this paper, we partially answered this question by characterizing total k-uniform chordal graphs.

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We believe that there exist connected total k-uniform graphs for every even k and finding such graphs is a topic of possible research