Domination Parameters in Graphs II

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Definitions

Roman domination

Roman dominating function

- Cockayne, Dreyer, Hedetniemi, and Hedetniemi in 2004 defined a *Roman dominating function* (RDF) on a graph $G = (V, E)$ to be a function $f : V \to \{0, 1, 2\}$ satisfying the condition that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. Roman domination in graphs, *Discrete Math.* **278** (2004).

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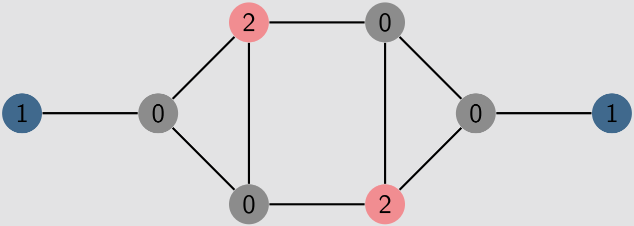

Definitions

Roman domination

Roman dominating function

- For a real–valued function, $f\colon V \to \mathbb{R}$, the *weight* of f is $\mathrm{w}(f) = \bigcup_{v \in V} f(v)$.
- The *Roman domination number*, denoted $\gamma_R(G)$, is the minimum weight of an RDF in G; that is, $\gamma_R(G) = \min\{\mathrm{w}(f) : f \text{ is a RDF in } G\}$.

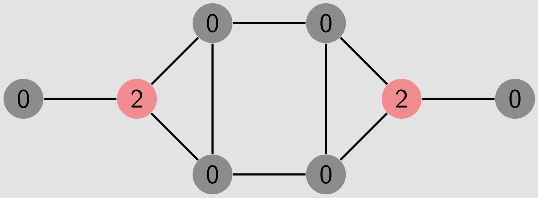

Figure: $\mathrm{w}(f) = 4$, $\gamma_R(G) = 4$

Roman graphs

Roman graphs

- Graphs which have Roman domination number equal to twice their domination number are called *Roman graphs*.
- The following classes of graphs were found to be Roman (Cockayne et al.):
  - P_{3k}, P_{3k+2} for $k \geq 1$,
  - C_{3k}, C_{3k+2} for $k \geq 1$,
  - $K_{m,n}$ for $\min\{m, n\} \neq 2$,
  - any graph with $\Delta(G) = n - 1$.
- Henning characterized Roman trees.

Roman trees

Henning, A characterization of Roman trees (2002)

- Let $\mathcal{F}_1^*$ denote the family of all rooted trees such that every leaf different from the root is at distance 2 from the root and all, except for possibly one, child of the root is a strong support vertex.

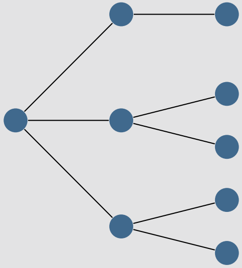

Figure: A tree from the family $\mathcal{F}_1^*$

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- Let $\mathcal{F}_2^*$ denote the family of all rooted trees such that every leaf is at distance 2 from the root and all but two children of the root are strong support vertices.

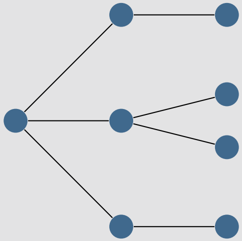

Figure: A tree from the family $\mathcal{F}_2^*$

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Henning, A characterization of Roman trees (2002)

- For a tree T, we let
  $V_S(T) = \{v \in V(T)\colon \cancel{v \in S(T)} \text{ and } \gamma_R(T-v) \geq \gamma_R(T)\}$.

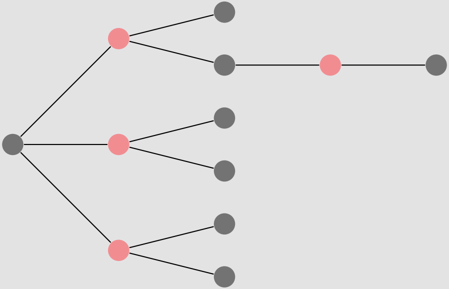

Figure: $\gamma_R(T) = 8$

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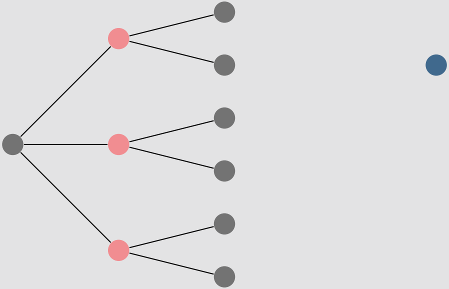

Figure: $\gamma_R(T - v) = 7$

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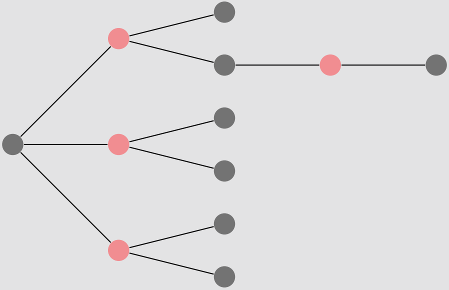

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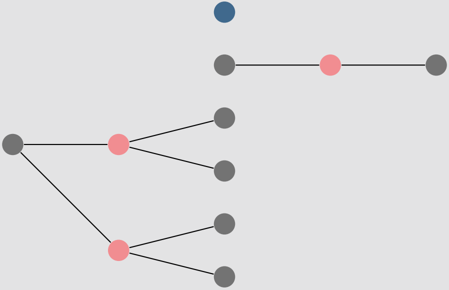

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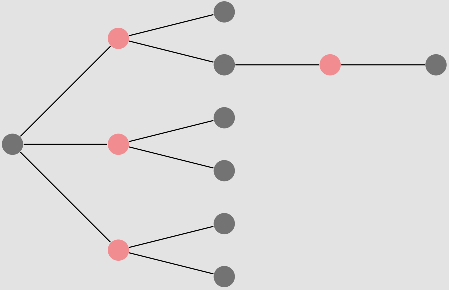

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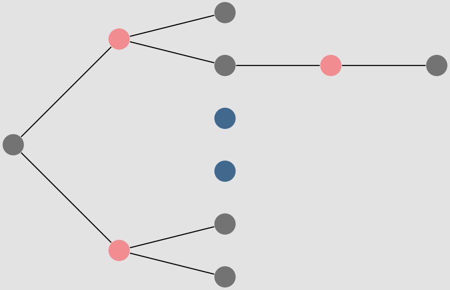

Figure: $\gamma_R(T - v) = 8$

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- Let $\mathcal{T}$ be the family of unlabelled trees T that can be obtained from a sequence $T_1, \ldots, T_j$ $(j \geq 1)$ of trees such that T_1 is a star $K_{1,r}$ for $r \geq 1$, and, if $j \geq 2$, T_{i+1} can be obtained recursively from T_i by one of the three operations $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$.

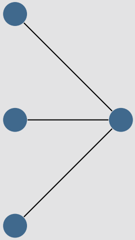

Figure: A tree T_1

Roman trees

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- **Operation** $\mathcal{T}_1$. Assume $x \in V_S(T_i)$. Then the tree T_{i+1} is obtained from T_i by adding a star $K_{1,s}$ for $s \geq 2$ with central vertex w and adding the edge xw.

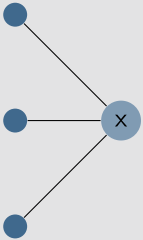

Figure: Operation $\mathcal{T}_1$

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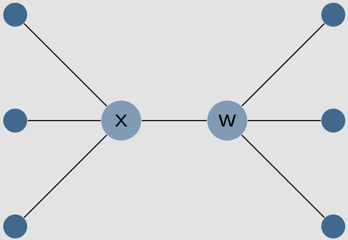

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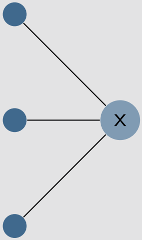
- **Operation** $\mathcal{T}_2$. Assume $x \in V(T_i)$. Then the tree T_{i+1} is obtained from T_i by adding a tree T from the family $\mathcal{F}_1^*$ and adding the edge xw, where w is a leaf of T if $T = P_3$ or w is the central vertex of T if $T \neq P_3$.



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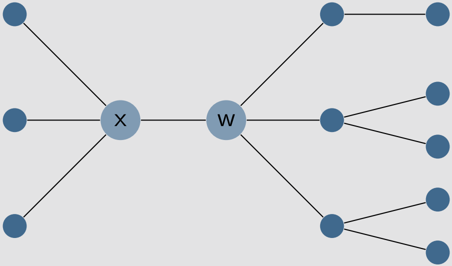

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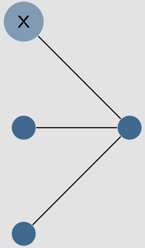

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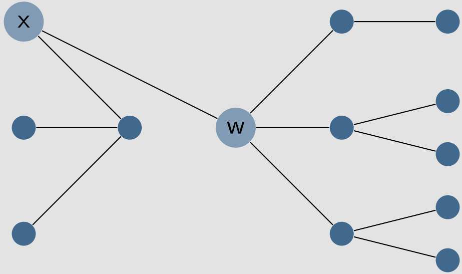

Figure: Operation $\mathcal{T}_2$

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- **Operation** $\mathcal{T}_3$. Assume $x \in V_S(T_i)$. Then the tree T_{i+1} is obtained from T_i by adding a tree T from the family $\mathcal{F}_2^*$ and adding the edge xw, where w denotes the central vertex of T.

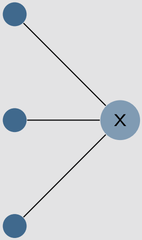

Figure: Operation $\mathcal{T}_3$

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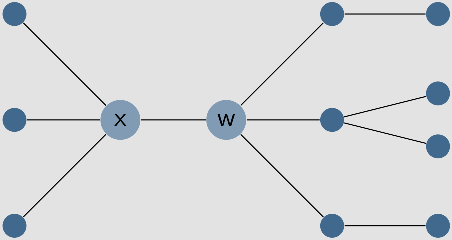

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Bernal, Zuazua (2018)

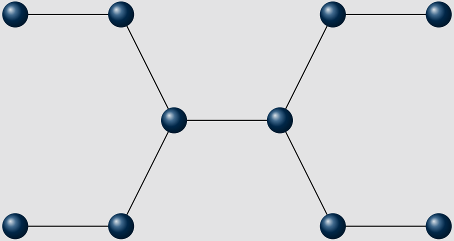

Figure: A $(\gamma_R, 2\gamma)$–tree not obtained by the original characterization of Henning

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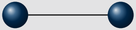

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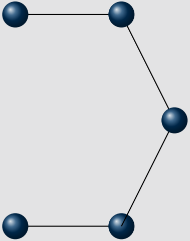

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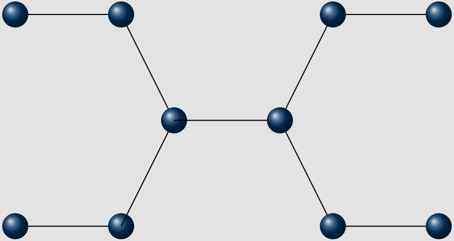

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Definitions

Weakly connected Roman domination

Weakly connected Roman dominating function

- We call the function f a *weakly connected Roman dominating function* in G (WCRDF) if each vertex $u \in V_0$ is adjacent to a vertex $v \in V_2$ and the subgraph $\langle V_1 \cup V_2 \rangle_w$ weakly induced by $V_1 \cup V_2$ is connected in G.

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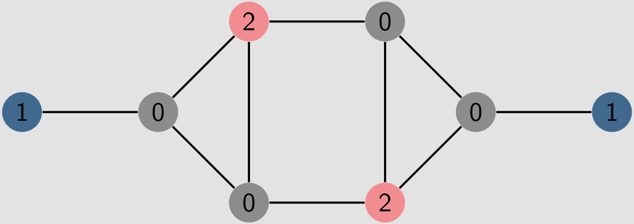

Definitions

Weakly connected Roman domination

Weakly connected Roman dominating number

- We define the *weight* $\mathrm{w}(f)$ of f to be $|V_1| + 2|V_2|$.
- The *weakly connected Roman domination number*, denoted $\gamma_R^{wc}(G)$, is the minimum weight of a WCRDF in G.
- A WCRDF of weight $\gamma_R^{wc}(G)$ we call a $\gamma_R^{wc}(G)$–function.

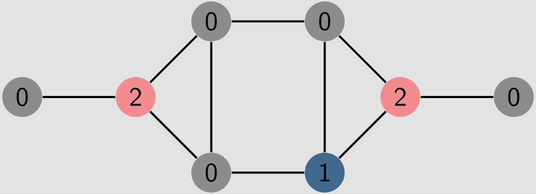

Figure: $\mathrm{w}(f) = 5, \gamma_R^{wc}(G) = 5$

Complexity results

Decision problem

WEAKLY CONNECTED ROMAN DOMINATING FUNCTION (WCRDF)
Instance: A connected graph and a positive integer k.
Question: Does G have a weakly connected Roman dominating function of weight at most k?

Complexity results

Theorem

WCRDS is NP-complete, even for chordal graphs.

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Polynomial transformation from EXACT COVER BY 3-SETS (X3C).

Paths

Theorem

For a path P_n on $n \geq 1$ vertices,

$$\gamma_R^{wc}(P_n) = \left\lfloor \frac{3}{4}n + \frac{1}{2} \right\rfloor.$$

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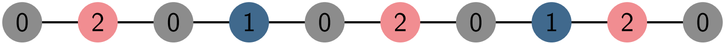

Figure: $\gamma_R^{wc}(P_{10}) = 8$

Lower bound on γ_R^{wc} of a tree without strong support vertices

Theorem

Let T be a tree without a strong support vertex. Then

$$\gamma_R^{wc}(T) \geq \left\lceil \frac{n}{2} \right\rceil + 1,$$

with equality if and only if T belongs to the family $\mathcal{T}$.

Lower bound on γ_R^{wc} of a tree without strong support vertices

Family $\mathcal{T}$

- Let $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$ be the following thee operations defined on a tree T.
- Let f be a $\gamma_R^{wc}(T)$– function and let $v \in V(T)$.

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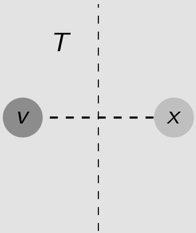
Operation $\mathcal{T}_1$. If $f(v) = 0$ and v is not a support vertex, then add a vertex x and the edge vx.



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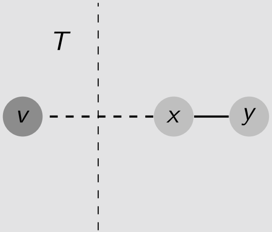
Operation $\mathcal{T}_2$. If $f(v) = 2$, add a path (x, y) and the edge vx.



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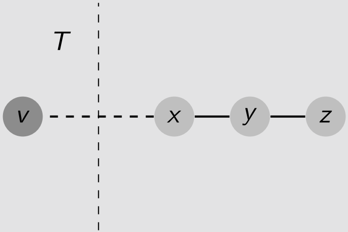
Operation $\mathcal{T}_3$. If $f(v) \in \{1, 2\}$, add a path (x, y, z) and the edge vx.



Lower bound on γ_R^{wc} of a tree without strong support vertices

Family $\mathcal{T}$

- Let $\mathcal{T}$ be the minimum family of trees obtained from P_2 by a finite sequence of Operations $\mathcal{T}_2$ and at most one either Operation $\mathcal{T}_1$ or $\mathcal{T}_3$.

Lower bound on γ_R^{wc} of a tree without strong support vertices

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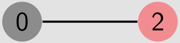

Figure: $\gamma_R^{wc}(T) = 2 = \lceil \frac{2}{2} \rceil + 1$

Lower bound on γ_R^{wc} of a tree without strong support vertices

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Figure: $\gamma_R^{wc}(T) = 3 = \left\lceil \frac{4}{2} \right\rceil + 1$

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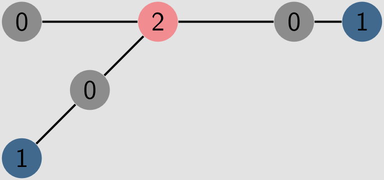

Figure: $\gamma_R^{wc}(T) = 4 = \left\lceil \frac{6}{2} \right\rceil + 1$

Lower bound on γ_R^{wc} of a tree without strong support vertices

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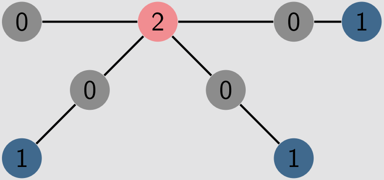

Figure: $\gamma_R^{wc}(T) = 5 = \left\lceil \frac{8}{2} \right\rceil + 1$

Lower bound on γ_R^{wc} of a tree without strong support vertices

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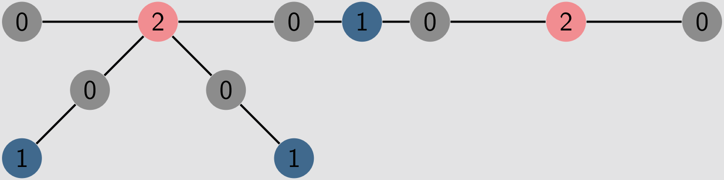

Figure: $\gamma_R^{wc}(T) = 7 = \lceil \frac{11}{2} \rceil + 1$

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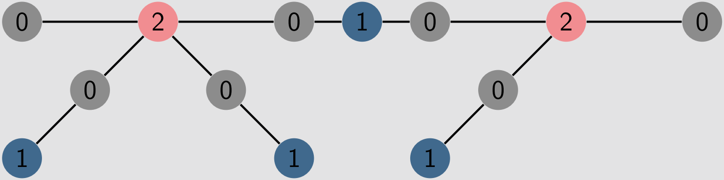

Figure: $\gamma_R^{wc}(T) = 8 = \left\lceil \frac{13}{2} \right\rceil + 1$

Lower bound on γ_R^{wc} of a tree without strong support vertices

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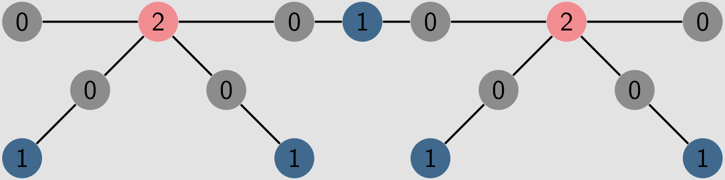

Figure: $\gamma_R^{wc}(T) = 9 = \left\lceil \frac{15}{2} \right\rceil + 1$

Lower bound on γ_R^{wc} of a tree without strong support vertices

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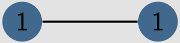

Figure: $\gamma_R^{wc}(T) = 2 = \left\lceil \frac{2}{2} \right\rceil + 1$

Lower bound on γ_R^{wc} of a tree without strong support vertices

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Figure: $\gamma_R^{wc}(T) = 4 = \left\lceil \frac{5}{2} \right\rceil + 1$

Lower bound on γ_R^{wc} of a tree without strong support vertices

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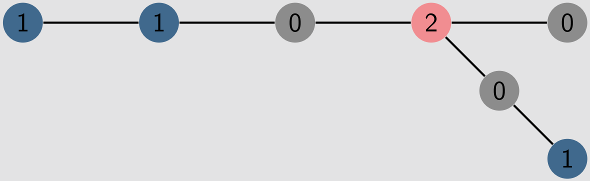

Figure: $\gamma_R^{wc}(T) = 5 = \left\lceil \frac{7}{2} \right\rceil + 1$

Lower bound on γ_R^{wc} of a tree without strong support vertices

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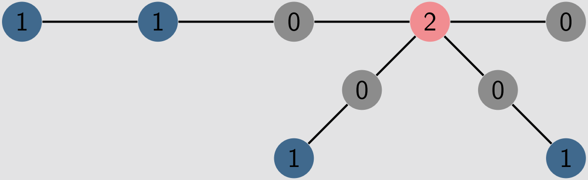

Figure: $\gamma_R^{wc}(T) = 6 = \left\lceil \frac{9}{2} \right\rceil + 1$

Lower bound on γ_R^{wc} of a tree without strong support vertices

Family $\mathcal{T}$

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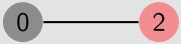

Figure: $\gamma_R^{wc}(T) = 2 = \lceil \frac{2}{2} \rceil + 1$

Lower bound on γ_R^{wc} of a tree without strong support vertices

Family $\mathcal{T}$

- Let $\mathcal{T}$ be the minimum family of trees obtained from P_2 by a finite sequence of Operations $\mathcal{T}_1$ and at most one either Operation $\mathcal{T}_2$ or $\mathcal{T}_3$.

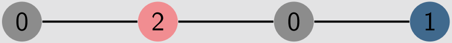

Figure: $\gamma_R^{wc}(T) = 3 = \left\lceil \frac{4}{2} \right\rceil + 1$

Lower bound on γ_R^{wc} of a tree without strong support vertices

Family $\mathcal{T}$

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Figure: $\gamma_R^{wc}(T) = 4 = \left\lceil \frac{5}{2} \right\rceil + 1$

Lower bound on γ_R^{wc} of a tree without strong support vertices

Corollary

Let T be a tree without a strong support vertex and with $\mathrm{diam}(T) \geq 9$*. Then*

$$\gamma_R^{wc}(T) \geq \left\lceil \frac{n}{2} \right\rceil + 2,$$

Upper bound on γ_R^{wc} of a tree

Theorem

If T is a tree of order $n \geq 3$, then

$$\gamma_R^{wc}(T) \leq \frac{5}{6}n,$$

with equality if and only if $T \in \mathcal{F}$.

Upper bound on γ_R^{wc} of a tree

Family $\mathcal{T}$

- Let $\mathcal{F}$ be a family of all trees T whose vertex set can be partitioned into sets, each set inducing a path P_6, such that the subgraph induced by the two central vertices of these P_6's is connected.

Upper bound on γ_R^{wc} of a tree

Family $\mathcal{T}$

- Let $\mathcal{F}$ be a family of all trees T whose vertex set can be partitioned into sets, each set inducing a path P_6, such that the subgraph induced by the two central vertices of these P_6's is connected.
- We call the subtree induced by these central vertices the underlying subtree of the resulting tree T, and we call each such path P_6 a base path of the tree T.

Upper bound on γ_R^{wc} of a tree

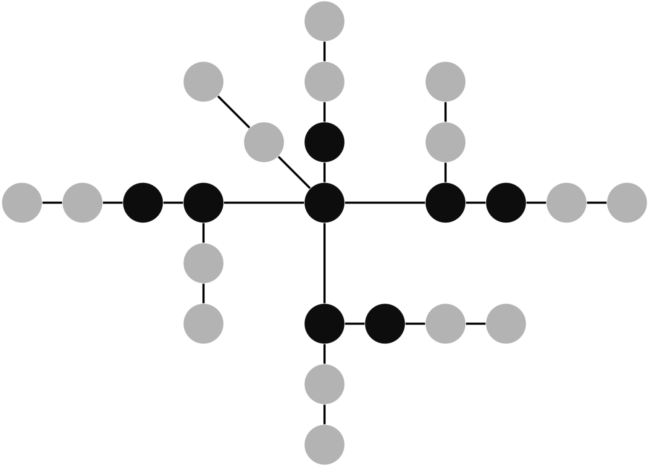
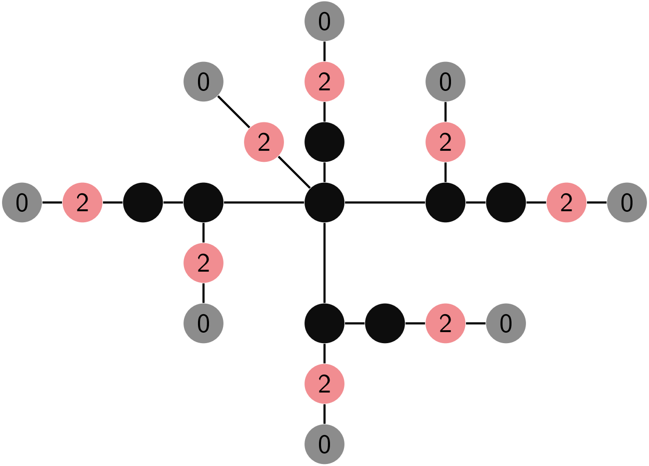


Figure: A tree in $\mathcal{F}$ with underlying tree denoted black

Upper bound on γ_R^{wc} of a tree



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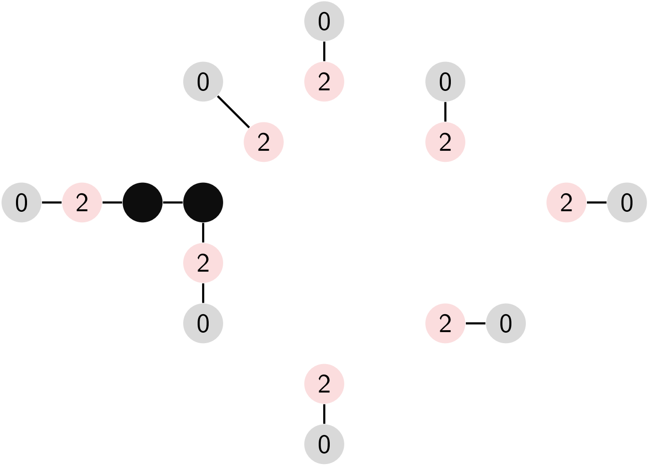
Upper bound on γ_R^{wc} of a tree



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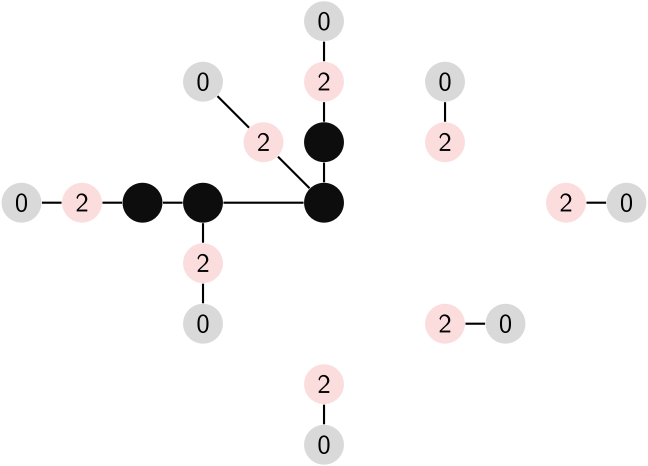

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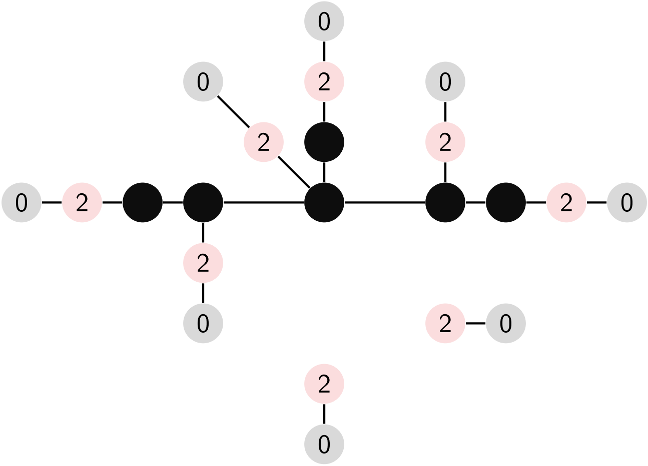
Upper bound on γ_R^{wc} of a tree



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Upper bound on γ_R^{wc} of a tree

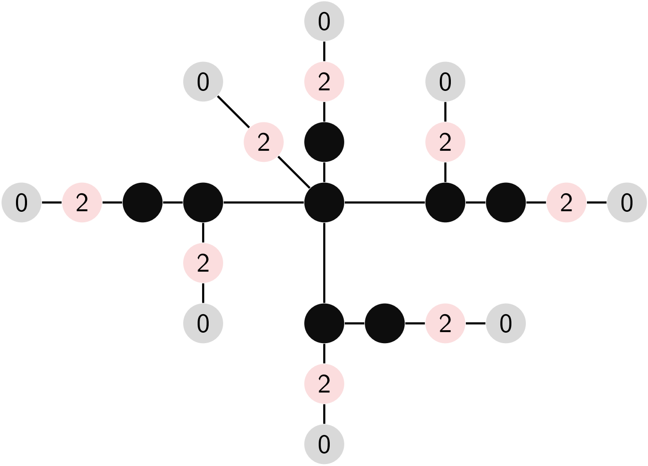

Figure: A tree in $\mathcal{F}$

γ_{wc}–excellent trees

- Domke, Hattingh, Marcus have defined the class $\mathcal{E}$ to be the class of trees obtained from P_2 by a finite sequence of the following operation: attach to any vertex a P_2.

γ_{wc}–excellent trees

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- A graph G is a γ_{wc}–excellent graph if each vertex of G is contained in some $\gamma_{wc}(G)$–set.

γ_{wc}–excellent trees

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Theorem (G.S. Domke, J.H. Hattingh, L.R. Marcus (2005))

A nontrivial tree T is γ_{wc}–excellent if and only if T belongs to the family $\mathcal{E}$.

γ_{wc}–excellent trees

Corollary (JR (2008))

Let T be a tree of order n at least 3. Then the following conditions are equivalent:

1. *T belongs to the family $\mathcal{E}$;*

γ_{wc}–excellent trees

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Corollary (JR (2008))

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γ_{wc}–excellent trees

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4. $\gamma_{wc}(T) = \frac{n}{2}$;

γ_{wc}–excellent trees

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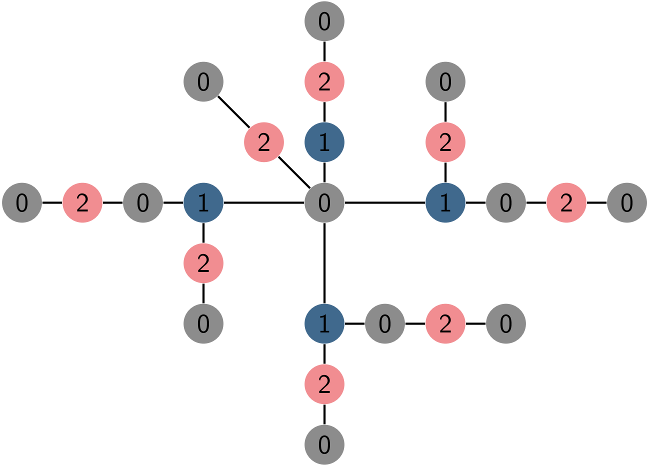
4. $\gamma_{wc}(T) = \frac{n}{2}$;
5. $sd_{\gamma_{wc}}(T) = 2$.

Upper bound on γ_R^{wc} of a tree



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Total Roman domination

H. Abdollahzadeh Ahangar, M.A. Henning, V. Samodivkin, I. Gonzalez Yero, *Total Roman Domination in Graphs*, Applicable Analysis and Discrete Math. (2016).

- A *total Roman domination function* is a Roman dominating function with the additional property that the subgraph of G induced by the set of all vertices of positive weight has no isolated vertex.

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- A *total Roman domination function* is a Roman dominating function with the additional property that the subgraph of G induced by the set of all vertices of positive weight has no isolated vertex.
- *The total Roman domination number* is the minimum weight of a total Roman domination function on G.

Weak Roman domination

S.T. Hedetniemi, M.A. Henning, *Defending the Roman Empire – A new strategy*, Discrete Math. 266 (2003) 239–251.

- A vertex u with $f(u) = 0$ is said to be *undefended* with respect to f, if it is not adjacent to a vertex with positive weight.

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- A vertex u with $f(u) = 0$ is said to be *undefended* with respect to f, if it is not adjacent to a vertex with positive weight.
- The function f is *a weak Roman dominating function* (WRDF) if each vertex u with $f(u) = 0$ is adjacent to a vertex v with $f(v) > 0$ such that the function $f' \colon V \to \{0, 1, 2\}$ defined by $f'(u) = 1$, $f'(v) = f(v) - 1$ and $f'(w) = f(w)$ if $w \in V - \{u, v\}$, has no undefended vertex.

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- The weight of f is $w(f) = \sum_{v \in V} f(v)$. The *weak Roman domination number*, denoted by $\gamma_r(G)$, is the minimum weight of a WRDF in G.
- For any graph G,

$$\gamma(G) \le \gamma_r(G) \le \gamma_R(G) \le 2\gamma(G).$$

Definitions

$(1, 2)$-domination

- A set $D \subseteq V(G)$ is a $(1, 2)$-dominating set if each vertex v of $V - D$ has a neighbour in D as well as another vertex of D at a distance not greater than 2 from v.

Definitions

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- The $(1,2)$-domination number, denoted by $\gamma_{1,2}(G)$, is the cardinality of a smallest $(1,2)$-dominating set of G.
- The $(1,2)$-domination in graphs is a special case of $(1,k)$-domination. Here we only deal with the case when $k = 2$, that is, secondary domination.

NPC

$(1, 2)$-DOMINATING SET
Instance: A graph G and a positive integer k.
Question: Does G have a $(1, 2)$-dominating set of size at most k?

Theorem

$(1, 2)$-*DOMINATING SET is NP-complete, even for split graphs.*

Proof

Proof.

- The reduction is from EXACT COVER BY 3-SETS (X3C).
- Instance: $X = \{x_1, \ldots, x_{3q}\}$ and $C = \{C_1, \ldots, C_m\}$ of X3C, where C_j are subsets of X of size $|C_j| = 3$ for $1 \leq j \leq m$. Assume that $m \geq 2$, since otherwise the answer is trivial.

$\square$

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- We construct a split graph G with vertices for each $x_i \in X$, for each $c_j \in C$ and with edges $x_i C_j$ for all $x_i \in C_j$ and edges so that the subgraph induced by $\{C_1, \ldots, C_m\}$ is a complete graph K_m. Let $k = q$.

□

Domination number and $(1,2)$-domination number

Theorem

If G is a graph with $\delta(G) > 1$ and without a triangle, then

$$\gamma(G) = \gamma_{(1,2)}(G).$$

Proof.

Let G be a graph with $\delta(G) > 1$ and without a triangle. Suppose $\gamma(G) < \gamma_{(1,2)}(G)$. Then... □

Domination number and $(1,2)$-domination number

Proposition

If a graph G is bipartite and without a leaf, then $\gamma(G) = \gamma_{(1,2)}(G)$.

Domination number and $(1,2)$-domination number

Theorem

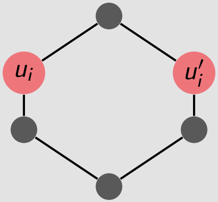
It is co-NP-hard to determine if for a given graph G there is $\gamma(G) = \gamma_{(1,2)}(G)$ even for bipartite graphs with only one leaf.

Proof.

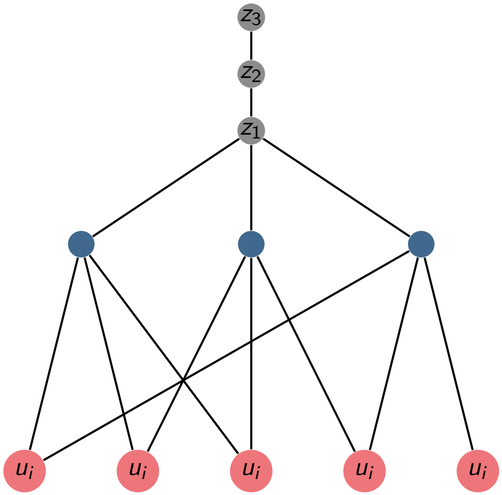
- Given an instance E, the set of literals $U = \{u_1, u_2, \ldots, u_n\}$ and the set of clauses $C = \{C_1, C_2, \ldots, C_m\}$ of 3SAT, we construct a bipartite graph G whose order is polynomially bounded in terms of n and m, and such that the formula is satisfiable if and only if $\gamma(G) < \gamma_{(1,2)}(G)$.

$\square$

For each literal u_i



For each clause C_j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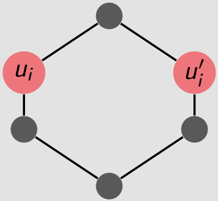
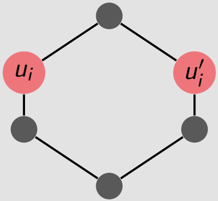
For each literal u_i

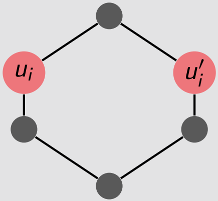


For each literal u_i

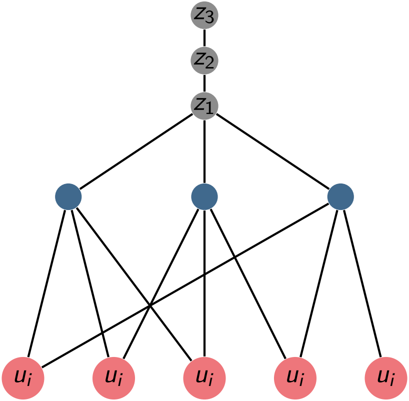


- At least two vertices of each C_6 belong to each $\gamma(G)$-set and to each $\gamma_{(1,2)}(G)$-set.

For each literal u_i



- At least two vertices of each C_6 belong to each $\gamma(G)$-set and to each $\gamma_{(1,2)}(G)$-set.
- Assume two vertices belong. If u_i belongs, then u'_i does not. If u'_i belongs, then u_i does not.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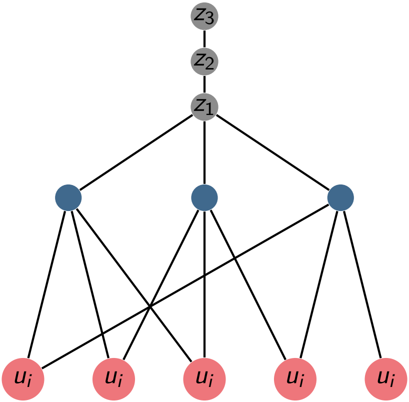
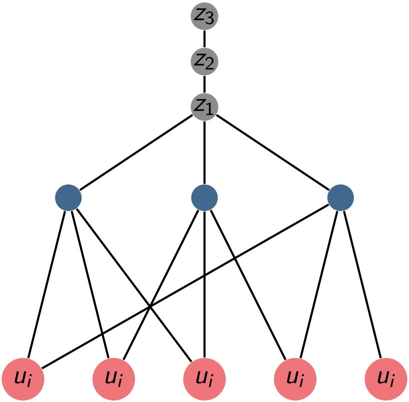
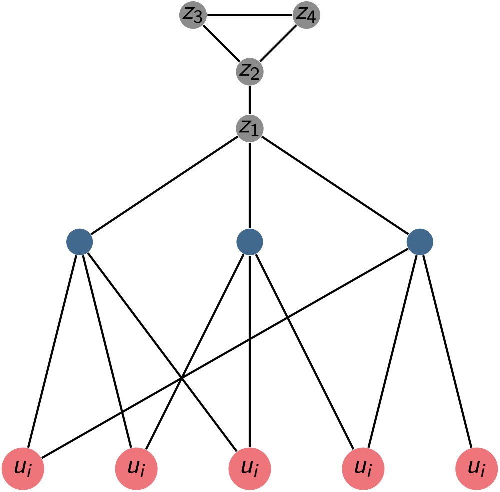


- $\gamma(G) \geq 2n+1$.



- $\gamma(G) \geq 2n+1.$
- $\gamma_{(1,2)}(G) = 2n+2.$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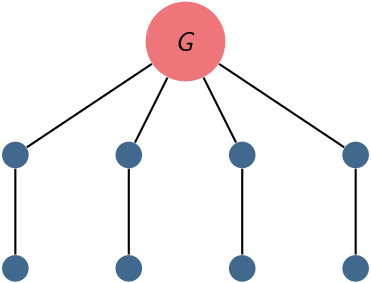
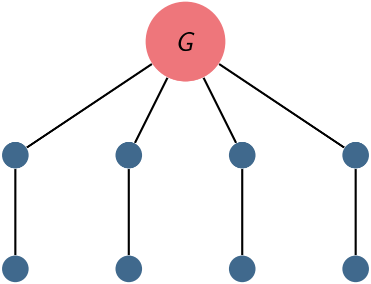
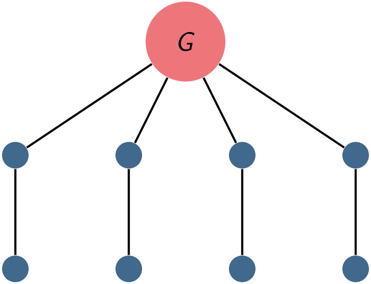
u_i

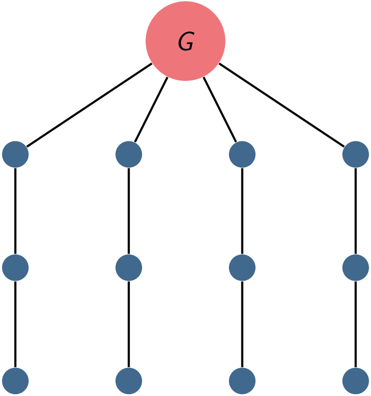

G

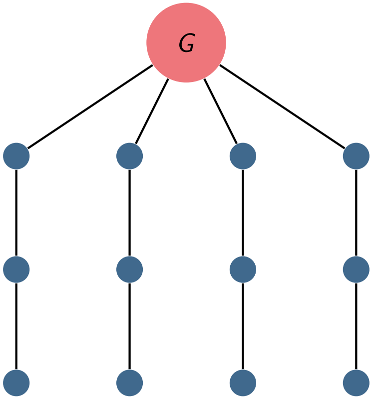


- $\gamma(H_1) = n(G)$.

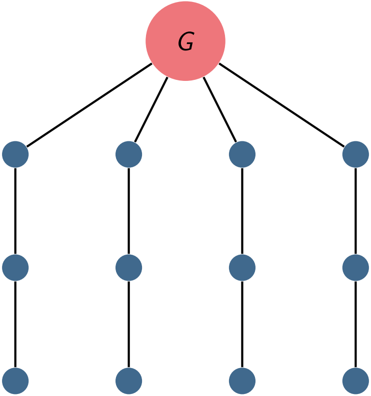


- $\gamma(H_1) = n(G)$.
- $\gamma_{(1,2)}(H_1) = n(G) + \gamma(G)$.


G



- $\gamma(H_2) = n(G) + \gamma(G)$.



- $\gamma(H_2) = n(G) + \gamma(G)$.
- $\gamma_{(1,2)}(H_2) = 2n(G)$.

Domination number and $(1,2)$-domination number

Theorem

There is a class of bipartite graphs for which determining the domination number is NP-hard and determining the $(1,2)$-domination number is polynomial and there is a class of bipartite graphs for which determining the domination number is polynomial and determining the $(1,2)$-domination number is NP-hard.

Domination number and $(1, 2)$-domination number

Theorem

$(1, 2)$-DOMINATING SET is NP-complete for chordal bipartite graphs, C_4-free graphs, maximum degree 4 graphs, partial grid graphs and planar graphs.

Proof.

□

Domination number and $(1,2)$-domination number

Theorem

$(1,2)$-DOMINATING SET is NP-complete for chordal bipartite graphs, C_4-free graphs, maximum degree 4 graphs, partial grid graphs and planar graphs.

Proof.

- Since $\gamma_{(1,2)}(H_1) = \gamma(G)$, the operation of obtaining H_1 from G is a polynomial reduction from an NP-complete problem of DOMINATING SET to the $(1,2)$-DOMINATING SET.

$\square$

Certified domination number

Certified dominating set

- A subset S of $V(G)$ is a *certified dominating set* of G if S is a dominating set and every vertex belonging to S has either zero or at least two neighbours in $V(G) - S$.
- The cardinality of a minimum certified dominating set in G is called the *certified domination number* of G and is denoted $\gamma_{cer}(G)$.

Certified domination

Theorem (M. Dettlaff, et. al., Graphs with equal domination and certified domination numbers)

If G is a graph in which $\delta(G) \geq 2$, then $\gamma(G) = \gamma_{cer}(G)$.

Certified domination

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If G is a graph in which $\delta(G) \geq 2$, then $\gamma(G) = \gamma_{cer}(G)$.

Theorem (JR (2019+))

It is NP-hard to determine if $\gamma_{cer}(G) \neq \gamma(G)$ even for graphs G with only one vertex of degree 1.

Certified domination

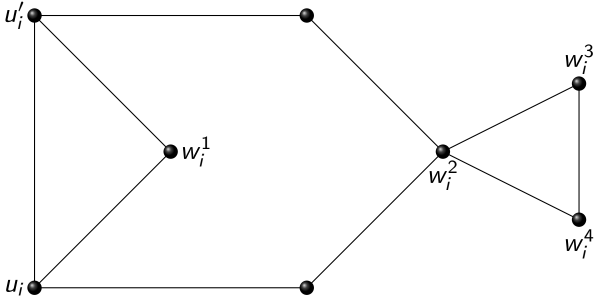


Figure: The graph $G(u_i)$

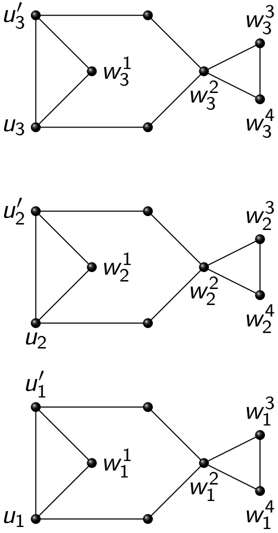


Figure: The edges between $G(C_1)$ and $G(u_1) \cup G(u_2) \cup G(u_3)$ for the clause $C_1 = \neg u_1 \vee u_2 \vee u_3$

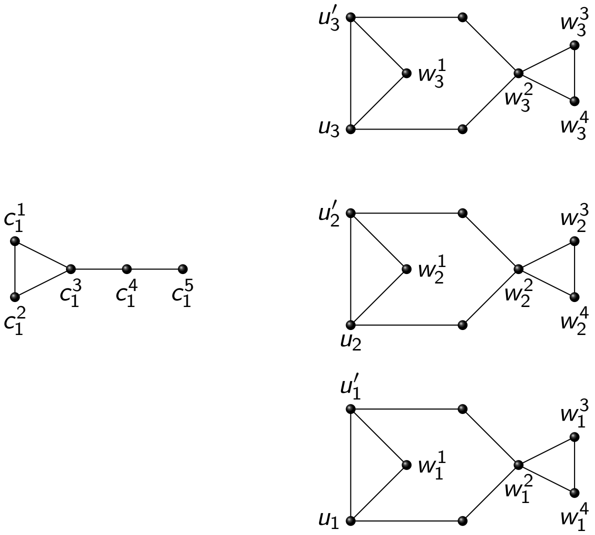


Figure: The edges between $G(C_1)$ and $G(u_1) \cup G(u_2) \cup G(u_3)$ for the clause $C_1 = \neg u_1 \vee u_2 \vee u_3$

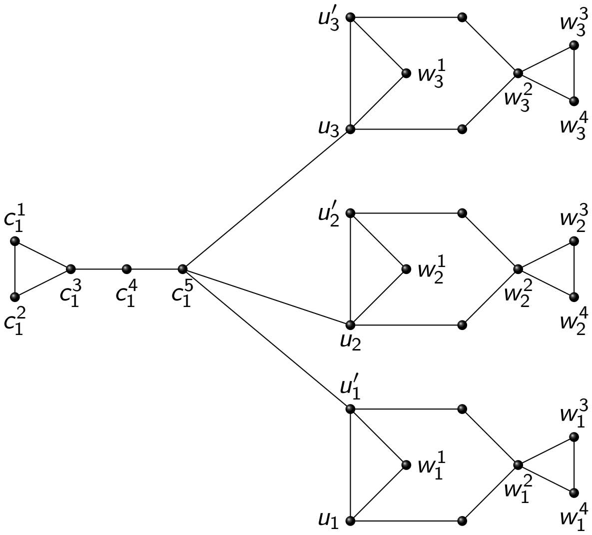
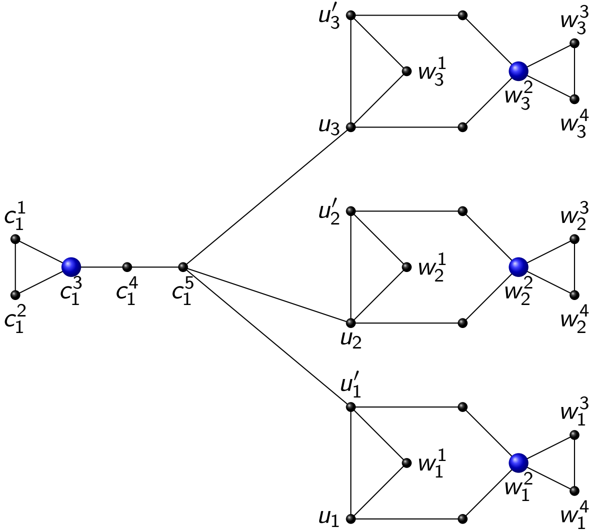
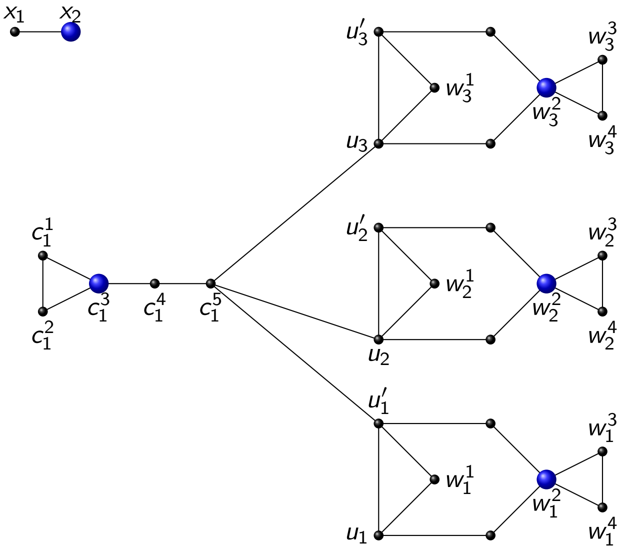
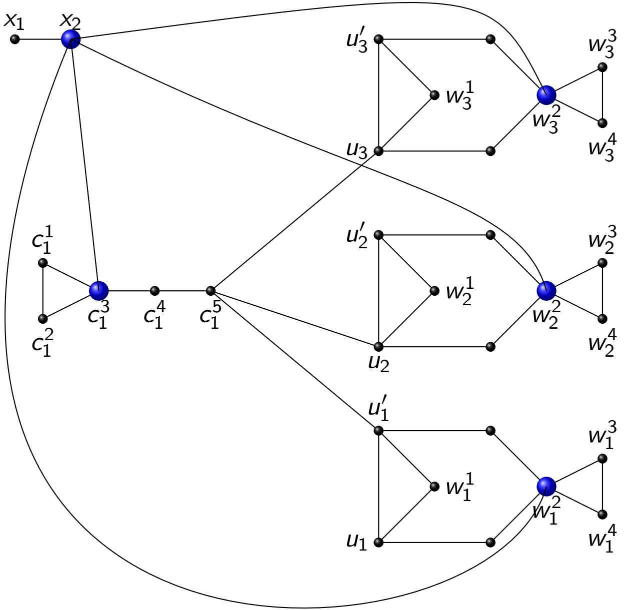


Figure: The edges between $G(C_1)$ and $G(u_1) \cup G(u_2) \cup G(u_3)$ for the clause $C_1 = \neg u_1 \vee u_2 \vee u_3$


u_3'
w_3^3
w_3^1
w_3^2
u_3
w_3^4
u_2'
w_2^3
c_1^1
w_2^1
c_1^3
c_1^4
c_1^5
w_2^2
c_1^2
w_2^4
u_2
u_1'
w_1^3
w_1^1
w_1^2
u_1
w_1^4


x_1
x_2
u_3'
w_3^1
w_3^2
w_3^3
w_3^4
u_3
c_1^1
c_1^2
c_1^3
c_1^4
c_1^5
u_2'
w_2^1
w_2^2
w_2^3
w_2^4
u_2
u_1'
w_1^1
w_1^2
w_1^3
w_1^4
u_1


x_1
x_2
u_3'
w_3^1
w_3^3
w_3^2
w_3^4
u_3
c_1^1
c_1^2
c_1^3
c_1^4
c_1^5
u_2'
w_2^1
w_2^3
w_2^2
w_2^4
u_2
u_1'
w_1^1
w_1^3
w_1^2
w_1^4
u_1

Thank you!