Extension of Dirac Theorem on Hamiltonicity *

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In this study, we focus on the classical Dirac theorem, which asserts that every graph on n vertices with minimum degree at least $\lceil n/2 \rceil$ is Hamiltonian. This lower bound of $\lceil n/2 \rceil$ on minimum degree of a graph strictly holds. In this paper, we extend the classical Dirac theorem by classifying the only graphs without a Hamiltonian cycle when the minimum degree is at least $\lfloor n/2 \rfloor$. Our proof is constructive and hence may lead to a polynomial time algorithm, which is left as a future work.

1 Introduction

A *Hamiltonian cycle* of a graph is a cycle which passes through every vertex of the graph exactly once, and a graph is *Hamiltonian* if it contains a Hamiltonian cycle. Finding a Hamiltonian cycle in a graph is one of the important problems in graph theory, and has been studied for years. Karp [4] proved that the problem of determining whether a Hamiltonian cycle exists in a given graph is NP-complete. However, in the past years some important sufficient conditions have been found. For instance, in [6] Ore proved that for all distinct nonadjacent pairs of vertices u, v of a graph G if the sum of degrees of u and v is at least the order of G, then G is Hamiltonian. One vital sufficient condition proved by Dirac [2] is that every graph on n vertices with minimum degree at least $\lceil n/2 \rceil$ is Hamiltonian. This lower bound on the minimum degree of a graph strictly holds, so no smaller minimum degree can be sufficient for hamiltonicity of a given graph in general. But we show that except two families of graphs this minimum degree bound can be lowered to $\lfloor n/2 \rfloor$.

Furthermore, some additional sufficient conditions have been found for special graph classes. Nash-Williams [5] proved that every k-regular graph on $2k + 1$ vertices is Hamiltonian. In [5], he also proved that a 2-connected graph of order n with minimum degree at least $\max\{(n + 2)/3, \beta\}$, where β is independence number, is Hamiltonian. Although this last result is more general than others, since finding the independence number of a graph is in general NP-hard, this result does not give an efficient algorithm to determine the hamiltonicity of any given graph. In contrast, here we give a constructive characterization without using independence

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number. We extend the classical Dirac theorem on Hamiltonian cycle to the case where the minimum degree is at least $\lfloor n/2 \rfloor$ by classifying non-Hamiltonian graphs under this condition.

We adopt [7] and [3] for terminology and notation not defined here. A graph $G = (V, E)$ is given by a pair of a vertex set $V = V(G)$ and a edge set $E = E(G)$. In this work, we consider only simple graphs, which have no loops or multiple edges. Particularly, G_n denotes not necessarily connected simple graph on n vertices. We use $|V(G)|$ to denote the order of G and $N(v)$ to denote the neighborhood of a vertex v of G. In addition, $d(v)$ denotes degree of a vertex v of G and $\delta(G)$ denotes the minimum degree of G. The *distance* $d(u, v)$ between two vertices u and v is the length of a shortest path joining u and v; and the *diameter* of G, denoted by $d(G)$, is the maximum distance among all pairs of vertices of G. Given a graph G with n vertices, the *closure* $cl(G)$ of G is uniquely constructed from G by repeatedly adding a new edge uv connecting a nonadjacent pair of vertices u and v with $d(v) + d(u) \geq n$ until no more pairs with this property can be found.

Moreover, there are binary operations which create a new graph from two initial graphs $G(V, E)$ and $G'(V', E')$. The *union* of two graphs $G(V, E)$ and $G'(V', E')$ is the union of their vertex and edge sets, denoted by $G \cup G' = (V \cup V', E \cup E')$. When V and V' are disjoint, their union is referred to as the *disjoint union*. The *join* of graphs G and G' is the disjoint union graph $G \cup G'$ together with all the edges joining V and V', denoted by $G + G'$. The complete graphs can be seen as the join of two complete graphs, $K_{n_1+n_2} = K_{n_1} + K_{n_2}$; and the complete bipartite graphs can be seen as the join of empty graphs, $K_{n,m} = \bar{K}_n + \bar{K}_m$.

The classical Dirac theorem, which we will extend as the main result of this work, is stated as follows:

Theorem 1. ***(Dirac [2])*** *If G is a graph of order $n \geq 3$ such that $\delta(G) \geq \lceil n/2 \rceil$, then G is Hamiltonian.*

We now present the main theorem of this paper:

Theorem 2. ***(Extension of Dirac Theorem)*** *Let G be a graph of order $n \geq 3$ such that $\delta(G) \geq \lfloor n/2 \rfloor$. Then G is Hamiltonian unless G is the graph $K_{\lceil n/2 \rceil} \cup K_{\lceil n/2 \rceil}$ with one common vertex or the graph $\bar{K}_{\lceil n/2 \rceil} + G_{\lfloor n/2 \rfloor}$ for odd n.*

2 Proof of the Main Theorem

In this section we constructively prove the main theorem. In contrast to Theorem 3, we extend the classical Dirac theorem to the case $\delta(G) \geq \lfloor n/2 \rfloor$ by classifying non-Hamiltonian graphs without using the independence number under this condition.

The following results will be used to establish the proof of the main theorem:

Lemma 3. ***(Nash-Williams [5])*** *Let G be a 2-connected graph of order n with $\delta(G) \geq \max\{(n + 2)/3, \beta\}$, where β is the independence number of G. Then G is Hamiltonian.*

Lemma 4. ***(Bondy-Chvátal [1])*** *A graph G is Hamiltonian if and only if its closure $cl(G)$ is Hamiltonian.*

We prove the main result of this work as follows:

Proof of Theorem 2. For $n = 2r$ where $r \in \mathbb{Z}^+$, the result holds by Theorem 1. Hence, we assume that $n = 2r + 1$ and $\delta(G) \geq r$. First, we add a vertex y to the graph G and connect it to all other vertices. This new graph G' has $|V(G')| = 2r + 2$ and $\delta(G') \geq r + 1$. By Theorem 1, it has a Hamiltonian cycle. After removing y, we still have a Hamiltonian path P in G, say $P = (x_0, ..., x_{2r})$.

Suppose G has no Hamiltonian cycle. That is, x_0 and x_{2r} are not adjacent. W.l.o.g, if $d(x_0) > r$ or $d(x_{2r}) > r$, then x_0 and x_{2r} are adjacent in closure of G. Hence, the closure $cl(G)$ is Hamiltonian and therefore G is Hamiltonian by Theorem 4. That is, we can assume that $d(x_0) = d(x_{2r}) = r$.

Now, we observe the following facts:

1. *If x_0 is adjacent to x_i, then x_{2r} is not adjacent to x_{i-1}.* Otherwise, the closed trail $x_0x_1..x_{i-1}x_{2r}x_{2r-1}x_{2r-2}..x_ix_0$ yields a Hamiltonian cycle.
2. *If x_0 is not adjacent to x_i, then x_{2r} is adjacent to x_{i-1}.* By the first fact, x_{2r} can be adjacent to vertices whose successive vertices in P are not adjacent to x_0. Since the number of vertices whose successive vertices in P are not adjacent to x_0 is r and $d(x_{2r}) = r$, x_{2r} has to be adjacent to the all such vertices.
3. *Every pair of non-adjacent vertices x_i and x_j have at least one common neighbor where $0 \leq i, j \leq 2r$.* Notice that the diameter $d(G) = 2$ since $|V(G)| = 2r+1$ and $d(x_i), d(x_j) \geq r$ for any $0 \leq i, j \leq 2r$.

Then, the following two cases arise:

Case 1: $N(x_0) \cup N(x_{2r}) = V(G)$ By assumption and the third fact, x_0 and x_{2r} have exactly one common neighbor x_k. Then x_{k-1} is not adjacent to x_{2r} but adjacent to x_0. Proceeding in the same way, we conclude that x_0 is adjacent to all vertices x_1 through x_k and x_{2r} is adjacent to all vertices x_k through x_{2r-1}. Since both vertices x_0 and x_{2r} have degree r, we conclude that $k = r$. Hence, there is an i_0 with $1 < i_0 \leq r$ such that x_{i_0} is adjacent to x_i for all $r+1 \leq i < 2r$. If there is a $x_{i_0} \neq x_r$ for any $r+1 \leq i < 2r$, then the cyle $x_{i_0}x_{i_0-1}..x_0x_{i_0+1}x_{i_0+2}..x_{i-1}x_{2r}x_{2r-1}..x_ix_{i_0}$ is a Hamiltonian cycle in G. If there is no $x_{i_0} \neq x_r$ for all $r+1 \leq i < 2r$, then we have a non-Hamiltonian graph $K_{\lceil n/2 \rceil} \cup K_{\lceil n/2 \rceil}$ with common vertex x_r.

Case 2: $N(x_0) \cup N(x_{2r}) \neq V(G)$ Then, there is an i_0 with $1 < i_0 < 2r - 1$ such that x_{i_0+1} is adjacent to x_0, but x_{i_0} is not. By the second fact, x_{i_0-1} must be adjacent to x_{2r}. Hence, we have a $(2r)$-cycle $x_{i_0-1}x_{i_0-2}..x_0x_{i_0+1}x_{i_0+2}..x_{2r}x_{i_0-1}$ not containing x_{i_0}. W.l.o.g, let $C = (y_1, y_2, ..., y_{2r})$ be the $(2r)$-cycle and y_0 be the remaining vertex. Note that C is a maximum cycle in G due to the assumption that G has no Hamiltonian cycle. It implies that y_0 cannot be adjacent to two consecutive vertices on C. Otherwise, C is not a maximum cycle and there exists a Hamiltonian cycle. Therefore, $d(y_0) = r$ and y_0 must be adjacent to every second vertex on C. W.l.o.g, let the second vertices on C be $y_1, y_3, ..., y_{2r-1}$. That is, y_0 is adjacent to all vertices with odd index and non-adjacent to any vertex with even index. Observe that replacing y_{2i} by y_0 gives another maximum cycle C' where $1 \leq i \leq r$, and then $d(y_{2i}) = r$ by the above argument on y_0. Therefore, every vertex with even index is adjacent to every vertex with odd index, and non-adjacent to any vertex with even index. Hence, we get the graph $\bar{K}_{\lceil n/2 \rceil} + G_{\lfloor n/2 \rfloor}$ where the vertices with even index form the empty graph $\bar{K}_{\lceil n/2 \rceil}$ and the vertices with odd index form a not necessarily connected graph $G_{\lfloor n/2 \rfloor}$. Notice that the graph $\bar{K}_{\lceil n/2 \rceil} + G_{\lceil n/2 \rceil}$ is a non-Hamiltonian graph since the order of $\bar{K}_{\lceil n/2 \rceil}$ is larger than the order of $G_{\lfloor n/2 \rfloor}$.

□

3 Conclusion

In this paper, we have extended the classical Dirac theorem to the case where the minimum degree is at least $\lfloor n/2 \rfloor$ by classifying non-Hamiltonian graphs under this condition. Our proof is constructive and hence, as a future work, we plan to design a polynomial time algorithm which produces a Hamiltonian cycle in a given graph satisfying our condition, if exists, or says it is non-Hamiltonian.

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