Discrete Applied Mathematics 252 (2019) 10–16

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

On one extension of Dirac's theorem on Hamiltonicity[☆]

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ARTICLE INFO

Article history:
Received 23 December 2015
Received in revised form 5 January 2017
Accepted 17 January 2017
Available online 9 February 2017

Keywords:
Hamiltonian graph
Sufficiency condition
The minimum degree
The Dirac's theorem
A self-contained constructive proof
Graph algorithms
A path
A cycle

ABSTRACT

The classical Dirac theorem asserts that every graph G on $n \geq 3$ vertices with minimum degree $\delta(G) \geq \lceil n/2 \rceil$ is Hamiltonian. The lower bound of $\lceil n/2 \rceil$ on the minimum degree of a graph is tight. In this paper, we extend the classical Dirac theorem to the case where $\delta(G) \geq \lfloor n/2 \rfloor$ by identifying the only non-Hamiltonian graph families in this case. We first present a short and simple proof. We then provide an alternative proof that is constructive and self-contained. Consequently, we provide a polynomial-time algorithm that constructs a Hamiltonian cycle, if exists, of a graph G with $\delta(G) \geq \lfloor n/2 \rfloor$, or determines that the graph is non-Hamiltonian. Finally, we present a self-contained proof for our algorithm which provides insight into the structure of Hamiltonian cycles when $\delta(G) \geq \lfloor n/2 \rfloor$ and is promising for extending the results of this paper to the cases with smaller degree bounds.

1. Introduction

A cycle passing through every vertex of a graph G exactly once is called a *Hamiltonian cycle* of G, and a graph containing a Hamiltonian cycle is called *Hamiltonian*. Finding a Hamiltonian cycle in a graph is a fundamental problem in graph theory and has been widely studied. In 1972, Karp [8] proved that the problem of determining whether a given graph is Hamiltonian is NP-complete. Hence, finding sufficient conditions for Hamiltonicity has been an interesting problem in graph theory.

Sufficient conditions for Hamiltonicity: The following Theorem proven in 1952 by Dirac provides an important sufficient condition for Hamiltonicity.

Theorem 1 (*[5]*). *If G is a graph of order $n \geq 3$ such that $\delta(G) \geq n/2$, then G is Hamiltonian.*

This lower bound on the minimum degree is tight; i.e., for every $k < n/2$, there is a non-Hamiltonian graph with minimum degree k. In 1960, Ore [14] proved that the following weaker condition is also sufficient for Hamiltonicity: if for every nonadjacent pair of vertices u and v of a graph G, the sum the of degrees of u and v is at least the order of G, then G is Hamiltonian. The *closure* of a graph G is obtained from G by repeatedly adding edges between pairs of non-adjacent vertices whose degree sum is at least the order of G. In 1976, Bondy and Chvátal [4] proved that even a weaker condition is sufficient: a graph G is Hamiltonian if and only if its closure is Hamiltonian.

☆ This work is supported by TUBITAK-CNRS under Grant No. 114E731.
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http://dx.doi.org/10.1016/j.dam.2017.01.011
0166-218X/

Some additional sufficient conditions have been found for special graph classes. In 1966, Nash-Williams proved the following:

Theorem 2 (*[12]*). *Every k-regular graph on* $2k+1$ *vertices is Hamiltonian.*

In 1971, he proved the following result that is stronger than the classical Dirac theorem.

Theorem 3 (*[13]*). *Let G be a 2-connected graph of order n with independence number* $\beta(G)$ *and minimum degree* $\delta(G)$*. If* $\delta(G) \geq \max((n+2)/3, \beta(G))$*, then G is Hamiltonian.*

Since finding the independence number of a graph is in general NP-hard, the above sufficiency condition cannot be tested in polynomial-time unless P = NP.

The Rahman–Kaykobad condition given in [15] is a relatively new condition that helps to determine the Hamiltonicity of a given graph G: The condition is that for every non-adjacent pair of vertices u, v of G, we have $d(u)+d(v)+dist(u,v) > |V|$, where $d(v)$ denotes the degree of v, $dist(u,v)$ denotes the length of a shortest path between u and v, and V denotes the set of vertices of G. In 2005, Rahman and Kaykobad [15] proved that a connected graph satisfying the Rahman–Kaykobad condition has a Hamiltonian path. In 2007, Mehedy et al. [11] proved that for a graph G without cut edges and cut vertices and satisfying the Rahman–Kaykobad condition, $dist(u,v) \geq 3$ and having a Hamiltonian path with endpoints u and v imply that G is Hamiltonian. In [9,10], it is proven that if G is a 2-connected graph of order $n \geq 3$ and $d(u)+d(v) \geq n-1$ for every pair of vertices u and v with $dist(u,v) = 2$, then G is Hamiltonian or a member of a given non-Hamiltonian graph class.

Toughness-related sufficient conditions for Hamiltonicity: Another important property of graphs related with Hamiltonicity is toughness. It is easy to see that being 1-tough is a necessary condition for Hamiltonicity. In 1990 Bauer, Hakimi and Schmeichel [2] proved that recognizing 1-tough graphs is NP-hard.

In 1978, Jung [7] proved that if G is a 1-tough graph on $n \geq 11$ vertices such that $d(x)+d(y) \geq n-4$ for every pair of non-adjacent vertices $x, y \in V(G)$, then G is Hamiltonian. In 1990, Bauer, Morgana and Schmeichel [3] provided a simple proof of Jung's theorem for graphs with more than 15 vertices. On the other hand, in 2002 Bauer et al. [1] presented a constructive proof of Jung's theorem for graphs on more than 15 vertices. Recognizing 1-tough graphs is in general NP-hard [2]. However, as a consequence of Jung's theorem, a graph G on $n \geq 11$ vertices is Hamiltonian if and only if G is 1-tough. It follows that when $\delta(G) \geq \frac{n}{2} - 2$, recognizing whether G is 1-tough can be solved in polynomial time [2].

Algorithmic results and our contribution: In 1992, Häggkvist [6] showed that for every positive integer k, the Hamiltonicity of a graph G on n vertices with $\delta(G) \geq n/2 - k$ can be determined in time $O(n^{5k})$.

In this paper, we first prove that a graph G with $\delta(G) \geq \lfloor n/2 \rfloor$ is Hamiltonian except two specific families of graphs. We first provide a simple proof using Nash-Williams theorem [13]. We then provide an alternative proof, which is simple, constructive, and self-contained. Using the constructive nature of our proof, we propose a polynomial-time algorithm that, given a graph G with $\delta(G) \geq \lfloor n/2 \rfloor$, constructs a Hamiltonian cycle of G, or says that G is non-Hamiltonian. Our algorithm can be used in any graph; however, if the input graph does not meet the degree condition $\delta(G) \geq \lfloor n/2 \rfloor$, the algorithm might fail to detect some Hamiltonian cycles. The main distinction of our work from [11] is that we propose a sufficient condition for Hamiltonicity by using condition $\delta(G) \geq \lfloor n/2 \rfloor$ and provide explicit non-Hamiltonian graph families, whereas [11] uses the Rahman–Kaykobad condition. Our proof also provides a novel insight into the pattern of vertices in a Hamiltonian cycle. We believe that this insight will play a pivotal role in extending our current results to a more general case. Notice that [9,10] show the same non-Hamiltonian graph classes as in our work. However, unlike [9,10], we obtain these graph classes constructively as a result of the nature of our proof. The main distinction of this work from [6] is that, [6] shows that Hamiltonicity can be determined in polynomial-time under such a minimum degree condition, whereas, in addition, we construct a Hamiltonian cycle (if exists) when $\delta(G) \geq \lfloor n/2 \rfloor$.

Recall that as a consequence of Jung's theorem, a graph G on $n \geq 11$ vertices is Hamiltonian if and only if G is 1-tough [2]. If $\delta(G) \geq \lfloor n/2 \rfloor$, a polynomial-time algorithm can then be designed by using the constructive proof of Bauer in [1], which either produces a Hamiltonian cycle or a set of vertices whose removal indicates that G is not 1-tough. However, our approach has the following advantages: (i) we specify non-Hamiltonian graph families under the minimum degree condition $\delta(G) \geq \lfloor n/2 \rfloor$, (ii) we explicitly provide a polynomial-time algorithm, (iii) we provide a shorter and simpler proof.

2. Preliminaries

We adopt [16] for terminology and notation not defined here. A graph $G = (V, E)$ is given by a pair of a vertex set $V = V(G)$ and an edge set $E = E(G)$, where $uv \in E(G)$ denotes an edge between two vertices u and v. In this work, we consider only simple graphs, i.e., graphs without loops or multiple edges. In particular, we use G_n to denote a simple graph on n vertices. $|V(G)|$ denotes the order of G and $N(v)$ denotes the neighbourhood of a vertex v of G. In addition, $\delta(G)$ denotes the minimum degree of G and the *distance* $dist(u,v)$ between two vertices u and v is the length of a shortest path joining u and v. The *diameter* of G, denoted by $diam(G)$, is the maximum distance among all pairs of vertices of G. If $P = x_0x_1x_2\dots x_k$ is a path, then we say that x_i *precedes* (resp. *follows*) x_{i+1} (resp. x_{i-1}) in P.

Given two graphs $G = (V, E)$ and $G' = (V', E')$, the *union* $G \cup G'$ of G and G' is the graph obtained by the union of their vertex and edge sets, i.e., $G \cup G' = (V \cup V', E \cup E')$. The *join* $G \vee G'$ of two disjoint graphs G and G' is obtained from their union by adding all edges joining V and V'. Formally, $G \vee G' = (V \cup V', E \cup E' \cup \{V \times V'\})$. G_n denotes a graph G on n vertices, while K_n and $\overline{K}_n$ denote the complete and empty graph, respectively, on n vertices.

We now present the main theorem of this paper:

Theorem 4. *Let G be a connected graph of order $n \geq 3$ such that $\delta(G) \geq \lfloor n/2 \rfloor$. Then G is Hamiltonian unless G is the graph $K_{\lceil n/2 \rceil} \cup K_{\lceil n/2 \rceil}$ with one common vertex or a graph $\overline{K}_{\lceil n/2 \rceil} \vee G_{\lfloor n/2 \rfloor}$ for odd n.*

The constructive nature of our proof for Theorem 4 given in Section 3 yields the following result that we prove in Section 4:

Theorem 5. *There is a polynomial-time algorithm that given a graph G of order $n \geq 3$ with $\delta(G) \geq \lfloor n/2 \rfloor$, determines whether G is Hamiltonian, and finds a Hamiltonian cycle in G, if such a cycle exists.*

3. Proofs of Theorem 4

In this section, we prove Theorem 4 that extends the classical Dirac theorem. The result is equivalent to Theorem 1 whenever n is even. Hence, we will prove for $n = 2r + 1$ for some $r \in \mathbb{Z}^+$, in which case $\delta(G) \geq \lfloor n/2 \rfloor = r$. We first provide a simple proof using Theorem 3.

Proof-1 of Theorem 4. First, we consider the case that G is not 2-connected. Let v be a cut vertex v, and $G^{(1)}, \ldots, G^{(k)}$ be the connected components of $G[V(G) \setminus \{v\}]$. Since a vertex of $G^{(i)}$ has at most one neighbour if $G \setminus G^{(i)}$ (namely v), we have $\left|V(G^{(i)})\right| - 1 \geq \delta(G^{(i)}) \geq r - 1$, thus $\left|V(G^{(i)})\right| \geq r$ for $i \in [1, k]$. Since $n = 2r + 1$, we have $k = 2$ and $\left|V(G^{(1)})\right| = \left|V(G^{(2)})\right| = r$. Therefore, $r - 1 = \left|V(G^{(i)})\right| - 1 \geq \delta(G^{(i)}) \geq r - 1$, implying that every vertex of $G^{(i)}$ is adjacent to every other vertex of $G^{(i)}$ and also to v, for $i \in \{1, 2\}$. Therefore, G is the graph $K_{\lceil n/2 \rceil} \cup K_{\lceil n/2 \rceil}$ with one common vertex.

Now consider the case that G is 2-connected. The only 2-connected graph on 3 vertices, namely K_3, is Hamiltonian; therefore, $n \geq 5$. If $n \geq 7$, then $\delta(G) \geq r = (n-1)/2 \geq (n+2)/3$. If $\delta(G) \geq \beta(G)$, then $\delta(G) \geq \max\{(n+2)/3, \beta(G)\}$ and G is Hamiltonian due to Theorem 3, a contradiction. Therefore, $\beta(G) > \delta(G) \geq r$ and hence $\beta(G) \geq r + 1$, i.e., G has an independent set S with $r + 1$ vertices. Since $\delta(G) \geq r$, every vertex of S is adjacent to every vertex of $V(G) \setminus S$, i.e., G is a graph $\overline{K}_{r+1} \vee G_r$, i.e. $\overline{K}_{\lceil n/2 \rceil} \vee G_{\lfloor n/2 \rfloor}$ as claimed.

For $n = 5$ consider the minimal graphs G' with $\delta(G') \geq 2$, i.e., those graphs G' that the removal of any edge violates the degree condition. By the minimality of G', the set U of vertices of degree more than 2 in G' is an independent set. Since $\delta(G') \geq 2$, we have $\left|V(G') \setminus U\right| \geq 3$, i.e., $|U| \leq 2$. If $|U| = 2$, then every vertex of U has to be adjacent to every vertex not in U so that its degree is 3. On the other hand, no two vertices in $V(G') \setminus U$ are adjacent since this would make their degrees at least 3. Therefore, G' is a $K_{2,3}$. If $U = \{u\}$, then $d(u)$ must be even by the handshaking lemma, i.e., $d(u) = 4$. Then, the degree sequence of $G' \setminus U$ is $(1, 1, 1, 1)$ and G' is a butterfly graph. Finally, if $U = \emptyset$, then G' is a cycle. We conclude that G is obtained by adding a (possibly empty) set of edges to a graph G' which is one of the following graphs: (i) a C_5, (ii) a butterfly, (iii) a $K_{2,3}$. If G' is a C_5, then G is clearly Hamiltonian. If G' is a butterfly, then G is obtained from it by the addition of at least one edge since a butterfly has a cut vertex. It is easy to verify that the addition of a single edge makes the butterfly Hamiltonian. If G' is a $K_{2,3}$, we observe that adding an edge to the bigger part of the bipartition makes the graph Hamiltonian. Therefore, G is either a $K_{2,3}$ or obtained from it by adding the only possible edge to the smaller part of the bipartition. Then G is a graph $\overline{K}_3 \vee G_2$ as claimed. □

We now present a self-contained, constructive and yet simple proof inspired by the proof of Theorem 2.

Proof-2 of Theorem 4. We start by considering the graph G' obtained by adding a new vertex y to G and connecting it to all other vertices. The graph G' has $2r + 2$ vertices and minimum degree at least $r + 1$. By Theorem 1, G' has a Hamiltonian cycle C. By the removal of y from C, we obtain a Hamiltonian path $P = x_0 x_1 \ldots x_{2r}$ of G.

Suppose that G has no Hamiltonian cycle. Then x_0 and x_{2r} are not adjacent. We observe the following facts:

1. *If x_0 is adjacent to x_i, then x_{2r} is not adjacent to x_{i-1}.* Otherwise, the closed trail $x_0 x_1 \ldots x_{i-1} x_{2r} x_{2r-1} x_{2r-2} \ldots x_i x_0$ is a Hamiltonian cycle.
2. *If x_0 is not adjacent to x_i, then x_{2r} is adjacent to x_{i-1}.* By Fact 1, $N(x_{2r}) \subseteq X = \{x_{i-1} | x_i \notin N(x_0), i \in [1, 2r]\}$. Since $|N(x_0)| \geq r$, we have $|X| \leq r$. Therefore, $r \leq |N(x_{2r})| \leq |X| \leq r$ and all inequalities must hold with equality. In particular, we have $N(x_{2r}) = X$, $d(x_{2r}) = r$ and $d(x_0) = r$.
3. *Every pair of non-adjacent vertices x_i and x_j, $i, j \in [0, 2r]$, has at least one common neighbour.* This is because $N(x_i) \subseteq V(G) \setminus \{x_i, x_j\}$, $N(x_j) \subseteq V(G) \setminus \{x_i, x_j\}$, $d(x_i) \geq r$, and $d(x_j) \geq r$. Note that this implies $diam(G) = 2$.

We now consider two disjoint and complementary cases:

1. $N(x_0) \cup N(x_{2r}) = V(G) \setminus \{x_0, x_{2r}\}$: By this assumption and Fact 3, x_0 and x_{2r} have exactly one common neighbour x_k. Then x_{k-1} is not adjacent to x_{2r} but adjacent to x_0. Proceeding in the same way, we conclude that $N(x_0) = \{x_1, \ldots, x_k\}$ and $N(x_{2r}) = \{x_k, \ldots, x_{2r-1}\}$. Since $d(x_0) = d(x_{2r}) = r$, we conclude that $k = r$. Let $i \in [r+1, 2r-1]$ and $i_0 \in [1, r-1]$. If $x_{i_0} x_i \in E$, the cycle $x_{i_0} x_{i_0-1} \ldots x_0 x_{i_0+1} x_{i_0+2} \ldots x_{i-1} x_{2r} x_{2r-1} \ldots x_i x_{i_0}$ is a Hamiltonian cycle of G. Therefore, for every $i \in [r+1, 2r-1]$ and every $i_0 \in [0, r-1]$, x_i and x_{i_0} are non-adjacent. Then $G = K_{\lceil n/2 \rceil} \cup K_{\lceil n/2 \rceil}$ with one common vertex x_r. Note that G is not Hamiltonian since it contains a cut vertex, namely x_r.

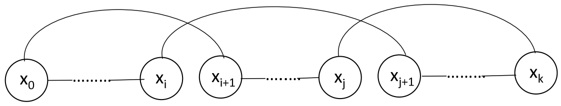

Fig. 1. The cycles detected by MakeTypeACycle.

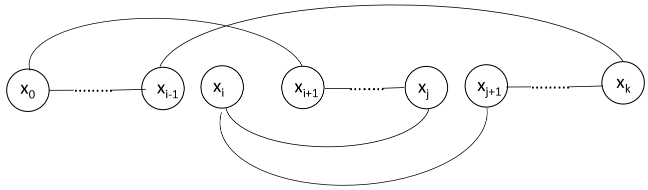

Fig. 2. The cycles detected by MakeTypeBCycle.

2. $N(x_0) \cup N(x_{2r}) \neq V(G) \setminus \{x_0, x_{2r}\}$: Then there is an $i_0 \in [2, 2r-2]$ such that x_{i_0+1} is adjacent to x_0, but x_{i_0} is not. By Fact 2, x_{i_0-1} is adjacent to x_{2r}. Hence, we have a $(2r)$-cycle $x_{i_0-1}x_{i_0-2} \ldots x_0 x_{i_0+1} x_{i_0+2} \ldots x_{2r} x_{i_0-1}$, which does not contain x_{i_0}, say C. We rename the vertices of C such that $y_1 y_2 \ldots y_{2r}$ and $y_0 = x_{i_0}$. If y_0 is adjacent to two consecutive vertices of C, then G is Hamiltonian. Therefore, y_0 is not adjacent to two consecutive vertices of C. Combining this with the fact that $d(y_0) \geq r$, we conclude that $d(y_0) = r$ and y_0 is adjacent to every second vertex of C. Without loss of generality, let $N(y_0) = \{y_1, y_3, \ldots, y_{2r-1}\}$. Observe that by replacing y_{2i} by y_0 for some $i \in [1, r]$, we obtain another cycle with $2r$ vertices. Then, by the same argument, $N(y_{2i}) = \{y_1, y_3, \ldots, y_{2r-1}\}$ for every $i \in [0, 2]$. Hence, $G = \overline{K}_{\lceil n/2 \rceil} \vee G_{\lfloor n/2 \rfloor}$, where the vertices with even index form the empty graph $\overline{K}_{\lceil n/2 \rceil}$ and the vertices with odd index form a not necessarily connected graph $G_{\lfloor n/2 \rfloor}$. Notice that G is not Hamiltonian since it contains an independent set with more than half of the vertices, namely $\{y_0, \ldots, y_{2r}\}$. □

In the following section, inspired by the above proof, we propose a polynomial-time algorithm to find a Hamiltonian cycle of a given graph G satisfying our minimum degree condition.

4. Proof of Theorem 5

In this section, we present Algorithm FindHamiltonian that, given a graph G, returns either a Hamiltonian cycle C or None. Although FindHamiltonian may in general return None for a Hamiltonian graph G, we will show that this will not happen if $\delta(G) \geq \lfloor n/2 \rfloor$. FindHamiltonian, whose pseudocode is given in Algorithm 1, first tests G for the two exceptional graph families mentioned in Theorem 4, i.e. graphs with vertex connectivity 1, and graphs G of the form $\overline{K}_{\lceil n/2 \rceil} \vee G_{\lfloor n/2 \rfloor}$ for odd n. In the latter case, $\bar{G}$ is the disjoint union of a $K_{\lceil n/2 \rceil}$ and a $G_{\lfloor n/2 \rfloor}$. These tests are done in lines 1 through 4. Once G passes the tests, the algorithm first builds a maximal path by starting with an edge and then extending it in both directions as long as this is possible. After this stage, the algorithm tries to find a larger path by closing the path to a cycle and then adding to it a new vertex and opening it back to a path. This is done in the main loop, in lines 6–14. Provided that MakeCycle never returns a cycle with less vertices than P, and since C can always be extended to a longer path in a connected graph, at least one of the following holds at the end of every iteration of the loop: $|V(C)| = n$, $C =$ None, the path P is strictly longer than in the beginning of the iteration. Since the graph is bounded, finally we will have either $|V(C)| = n$ or $C =$ None, in which case the loop terminates and returns C, which is either None or a Hamiltonian cycle of G. It remains to show that under the conditions of Theorem 4, i.e., whenever $\delta(G) \geq \lfloor n/2 \rfloor$, MakeCycle will always be able to construct a cycle from the vertices of P. MakeCycle tries three different constructions using the functions MakeTypeACycle (see Fig. 1), MakeTypeBCycle (see Fig. 2), and MakeTypeCCycle (see Fig. 3).

Note that Algorithm 1 is polynomial since (i) lines 1–4 can be computed in polynomial time, (ii) constructing a maximal path in lines 7–8, constructing a cycle in line 9, and obtaining a larger maximal path in lines 10–13 can be done in polynomial time, (iii) the loop in lines 6–14 iterates at most n times. Therefore, it is sufficient to prove the following lemma.

Lemma 6. *Let $|V(G)| = n \geq 3$, $\delta(G) \geq \lfloor n/2 \rfloor$ and P be a maximal path of G. If function* MakeCycle *returns* None*, then G has either a cut vertex or an independent set with more than $n/2$ vertices constituting a connected component of $\bar{G}$.*

Proof. Let $P = x_0 x_1 \ldots x_k$ be a maximal path of G for some $k \leq n-1$. Assume that the functions MakeTypeACycle, MakeTypeBCycle and MakeTypeCCycle all return None. Since P is maximal, $N(x_0), N(x_k) \subseteq V(P)$. Suppose that $x_0 x_k \in E(G)$. Then, setting $i = 0$ and $j = k-1$ in function MakeTypeACycle would detect a cycle. Therefore, $x_0 x_k \notin E(G)$, i.e., $N(x_0), N(x_k) \subseteq V(A) = \{x_1, \ldots, x_{k-1}\}$, where A is the path obtained by deleting the endpoints x_0 and x_k of P. We partition $V(A)$ by the adjacency of their vertices to x_0 and x_k. We denote the set of vertices $N(x_0) \setminus N(x_k)$ by A_0, $N(x_k) \setminus N(x_0)$ by A_k, $N(x_0) \cap N(x_k)$

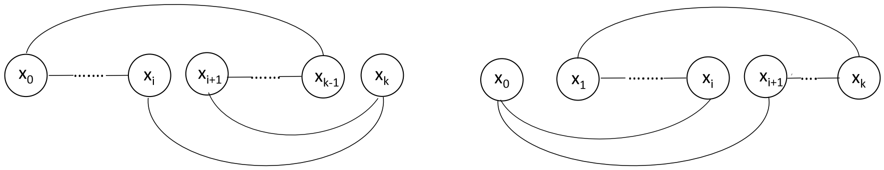

Fig. 3. The cycles detected by MakeTypeCCycle.

by A_{0k}, and the set of vertices $A \setminus (N(x_0) \cup N(x_k))$ by $A_{\overline{0k}}$. In the sequel, we use a_p to denote an arbitrary element of A_p for $p \in \{0, k, 0k, \overline{0k}\}$, and we use regular expression notation for sequences of elements of these sets. In particular, $(p)^*$ denotes zero or more repetitions of the pattern p.

Suppose that $x_i \in N(x_k)$ and $x_{i+1} \in N(x_0)$ for some $x_i \in V(A)$. Then, for this value of i and for $j = k-1$, the function MakeTypeACycle would detect a cycle. We conclude that such a vertex x_i does not exist in A. Therefore, two consecutive vertices (x_i, x_{i+1}) of A do not follow any of the following forbidden patterns: (a_k, a_0), (a_k, a_{0k}), (a_{0k}, a_0), (a_{0k}, a_{0k}). This is true since a pair (x_i, x_{i+1}) following one of these patterns implies $x_i \in N(x_k)$ and $x_{i+1} \in N(x_0)$.

Consider two vertices $x_i, x_j \in A_{0k}$ $(i < j)$, with no vertices from A_{0k} between them in A. Furthermore, suppose that there are no vertices from $A_{\overline{0k}}$ between x_i and x_j. By these assumptions and due to the forbidden pairs previously mentioned, we have $x_{i+1}, \ldots, x_{j-1} \in A_k$. However, (x_{j-1}, x_j) is also a forbidden pair, contradiction. Therefore, there is at least one vertex from $A_{\overline{0k}}$ between any two vertices of A_{0k}. We conclude that $\left|A_{\overline{0k}}\right| \geq |A_{0k}| - 1$. We have

$$\begin{aligned} n-2 \geq k-1 &= |A_{0k}| + |A_k| + |A_0| + \left|A_{\overline{0k}}\right| = (|A_{0k}| + |A_k|) + (|A_{0k}| + |A_0|) + \left|A_{\overline{0k}}\right| - |A_{0k}| \\ &\geq d(x_k) + d(x_0) - 1 \geq 2\delta(G) - 1 \\ \frac{n-1}{2} &\geq \delta(G). \end{aligned}$$

Since $\delta(G) \geq \lfloor n/2 \rfloor \geq \frac{n-1}{2}$, we have $\delta(G) = \frac{n-1}{2}$, and all the inequalities above hold with equality, implying the following:

(a) $d(x_0) = d(x_k) = \delta(G) = \frac{n-1}{2}$, thus n is odd and $|A_k| = |A_0|$.
(b) $k = n-1$, thus $V(P) = V(G)$.
(c) $\left|A_{\overline{0k}}\right| = |A_{0k}| - 1$. There is exactly one vertex of $A_{\overline{0k}}$ between two consecutive vertices from A_{0k} and there are no other vertices from $A_{\overline{0k}}$ in A.

The vertices between (and including) two consecutive vertices from A_{0k} follow the pattern $(a_{0k}a_k^*a_{\overline{0k}}a_0^*a_{0k})$. All vertices before the first vertex from A_{0k} are from A_0, and all vertices after the last vertex from A_{0k} are from A_k. We conclude that A follows the pattern:

$$a_0^*(a_{0k}a_k^*a_{\overline{0k}}a_0^*)^*a_{0k}a_k^*.$$

Then, every vertex of $A_{\overline{0k}}$ is preceded by a neighbour of x_k and followed by a neighbour of x_0; in other words, a vertex $x_i \in A_{\overline{0k}}$ satisfies the condition in Line 4 of MakeTypeBCycle. Since, because of our assumption, MakeTypeBCycle does not close a cycle, the condition in Line 6 is not satisfied for any value of j. We conclude that x_i is not adjacent to two consecutive vertices of A. Then, the number of neighbours of x_i among $x_1, \ldots, x_{i-1}$ is at most $\lceil \frac{i-1}{2} \rceil$ and the number of neighbours of x_i among $x_{i+1}, \ldots, x_{k-1}$ is at most $\lceil \frac{k-1-i}{2} \rceil$. Therefore,

$$d(x_i) \leq \left\lceil \frac{i-1}{2} \right\rceil + \left\lceil \frac{k-1-i}{2} \right\rceil \leq \frac{i}{2} + \frac{k-i}{2} = \frac{k}{2} = \frac{n-1}{2} = \delta(G).$$

Since $d(x_i) \geq \delta(G)$, all the inequalities above hold with equality, implying the following:

1. Both i and k are even
2. $N(x_i) = A_{odd}$ where $A_{odd} = \{x_1, x_3, \ldots, x_{k-1}\}$.

Since for every $x_i \in A_{\overline{0k}}$, x_i is even and $N(x_i) = A_{odd}$, we conclude that $A_{\overline{0k}}$ is an independent set. Recalling that $x_0x_k \notin E(G)$ and the definition of $A_{\overline{0k}}$, we conclude that $I = A_{\overline{0k}} \cup \{x_0, x_k\}$ is an independent set.

We observe in the previous pattern that the set of vertices preceding the neighbours of x_0 (i.e., $A_0 \cup A_{0k}$) is $A_0 \cup A_{\overline{0k}}$, and the set of vertices following the neighbours of x_k (i.e., $A_k \cup A_{0k}$) is $A_k \cup A_{\overline{0k}}$. Let x_i be a vertex that precedes a neighbour of x_0 and let x_j be a vertex that follows a neighbour of x_k with $i < j$. If $x_ix_j \in E$, MakeTypeACycle can close a cycle since the condition in Line 5 is satisfied. Therefore, a pair of adjacent vertices (x_i, x_j) with $i < j$ in G cannot follow one of the following patterns: $(a_{\overline{0k}}, a_{\overline{0k}})$, $(a_{\overline{0k}}, a_k)$, $(a_0, a_{\overline{0k}})$, (a_0, a_k). If $\left|A_{\overline{0k}}\right| = 0$, then $|A_{0k}| = 1$ and A follows the pattern $a_0^*a_{0k}a_k^*$. Since (a_0, a_k) is a forbidden pattern for adjacent vertices, none of the vertices of A_0 is adjacent to a vertex in A_k. Therefore, the unique vertex $a_{0k} \in A_{0k}$ is a cut vertex of G, contradicting our assumption. We conclude that $\left|A_{\overline{0k}}\right| > 0$.

Let $x_i \in A_{\overline{0k}}$ and $x_j \in A_{odd} = N(x_i)$. If $j < i$ then $x_j \notin A_0$, since otherwise they follow the pattern $(a_0, a_{\overline{0k}})$ and they are adjacent. Similarly, if $i < j$, then $x_j \notin A_k$. We conclude that, in A all the vertices between two vertices from $A_{\overline{0k}}$ are from A_{0k}. Moreover, all vertices before the first (after the last) vertex from $A_{\overline{0k}}$ except one vertex from A_{0k} are from A_k (resp. A_0). Then A follows the pattern:

$$a_{0k}a_k^*(a_{\overline{0k}}a_{0k})^*a_{\overline{0k}}a_0^*a_{0k}.$$

We now observe that $x_{k-1} \in A_{0k}$, i.e., $x_0x_{k-1} \in E(G)$. Let $\eta = |A_k| = |A_0|$. Suppose that $\eta \neq 0$. Then $x_kx_1, x_kx_2 \in E(G)$ and MakeTypeCCycle will close a cycle. Therefore, $\eta = 0$, i.e., A follows the pattern:

$$(a_{0k}a_{\overline{0k}})^*a_{0k}.$$

We conclude that I has $|I| = \frac{n+1}{2}$ vertices, and every vertex of I is adjacent to every vertex of $A_{odd} = A_{0k} = V(G) \setminus I$. Then I is a connected component of $\overline{G}$. □

Algorithm 1 FindHamiltonian

Require: A graph G with $|V(G)| = n$ and $\delta(G) \geq \lfloor n/2 \rfloor$
Ensure: C is a cycle of G

if G has a cut vertex **then return** None.
$\bar{G} \leftarrow$ the complement of G.
$\bar{H} \leftarrow$ the biggest connected component of $\overline{G}$.
if $\bar{H}$ is a complete graph, and $\left|V(\bar{H})\right| > \frac{n}{2}$ **then return** None.
$P \leftarrow$ a trivial path (a vertex) of G.
repeat
  while P is not maximal **do**
    Append an edge to P the get a longer path. ▷ P is a maximal path in G.
  $C \leftarrow$ MakeCycle(G, P).
  if $C \neq$ None and $|V(C)| \neq n$ **then**
    Let e be an edge with exactly one endpoint in C.
    Let e' be an edge of C incident to e. ▷ There are two such edges.
    $P \leftarrow C + e - e'$.
until $|V(C)| = n$ or $C =$ None
return C. ▷ C is a Hamiltonian cycle of G.

Algorithm 2 Making a Cycle

function MakeCycle(G, P)
Require: P is a maximal path in G.
Ensure: return a cycle C such that $V(C) = V(P)$ or None
  $C \leftarrow$ MakeTypeACycle(G, P).
  if $C \neq$ None **then return** C.
  $C \leftarrow$ MakeTypeBCycle(G, P).
  if $C \neq$ None **then return** C.
  $C \leftarrow$ MakeTypeCCycle(G, P).
  return C. ▷ Possibly $C =$ None.

Algorithm 3 Making a Type-A Cycle

function MakeTypeACycle(G, P)
Require: P is a maximal path in G.
Ensure: return a cycle C such that $V(C) = V(P)$ or None
  Let $P = x_0x_1 \ldots x_k$.
  for $i \in [0, k-3]$ **do**
    for $j \in [i+2, k-1]$ **do**
      if $x_0x_{i+1} \in E(G)$ and $x_ix_{j+1} \in E(G)$ and $x_jx_k \in E(G)$ **then**
        return $C = (x_0, x_1, \ldots, x_i, x_{j+1}, x_{j+2}, \ldots, x_k, x_j, x_{j-1} \ldots, x_{i+1}, x_0)$.
  return None.

Algorithm 4 Making a Type-B Cycle

function MakeTypeBCycle(G, P)
Require: P is a maximal path in G.
Ensure: return a cycle C such that $V(C) = V(P)$ or None
  Let $P = x_0x_1 \ldots x_k$.
  for $i \in [1, k-2]$ **do**
    if $x_0x_{i+1} \in E(G)$ and $x_{i-1}x_k \in E(G)$ **then**
      for $j \in [1, k-1] \setminus \{i\}$ **do**
        if $x_ix_j \in E(G)$ and $x_ix_{j+1} \in E(G)$ **then**
          return $C = (x_0, x_1, \ldots, x_{i-1}, x_k, x_{k-1}, \ldots, x_{j+1}, x_i, x_j, x_{j-1} \ldots, x_{i+1}, x_0)$.
  return None.

Algorithm 5 Making a Type-C Cycle

function MAKETYPECCYCLE(G, P)
Require: P is a maximal path in G.
Ensure: return a cycle C such that $V(C) = V(P)$ or NONE
  Let $P = x_0x_1 \dots x_k$.
  if $x_0x_{k-1} \in E(G)$ **then**
    for $i \in [0, k-2]$ **do**
      if $x_kx_i \in E(G)$ and $x_kx_{i+1} \in E(G)$ **then**
        return $C = (x_0, x_1, \dots, x_i, x_k, x_{i+1}, \dots, x_{k-1}, x_0)$
  if $x_kx_1 \in E(G)$ **then**
    for $i \in [1, k-1]$ **do**
      if $x_0x_i \in E(G)$ and $x_0x_{i+1} \in E(G)$ **then**
        return $C = (x_1, x_2, \dots, x_i, x_0, x_{i+1}, \dots, x_k, x_1)$
  return NONE.

5. Conclusion

In this work, we presented an extension of the classical Dirac theorem to the case where $\delta(G) \geq \lfloor n/2 \rfloor$. We identified the only non-Hamiltonian graph families under this minimum degree condition. Our proof is short, simple, constructive, and self-contained. Then, we provided a polynomial-time algorithm that constructs a Hamiltonian cycle, if exists, of a graph G with $\delta(G) \geq \lfloor n/2 \rfloor$, or determines that the graph is non-Hamiltonian. The proof we present for the algorithm provides insight into the pattern of vertices on Hamiltonian cycles when $\delta(G) \geq \lfloor n/2 \rfloor$. We believe that this insight will be useful in extending the results of this paper to graphs with lower minimum degrees, i.e., in identifying the exceptional non-Hamiltonian graph families when the minimum degree is smaller and constructing the Hamiltonian cycles, if exists. A natural question to ask in this direction is: What are the exceptional non-Hamiltonian graph families when $\delta(G) \leq \lfloor (n-1)/2 \rfloor$ or $\delta(G) \leq (n-2)/2$? How can we design an algorithm that not only determines whether a Hamiltonian cycle exists in such a case, but also constructs one if it exists? The investigation of these questions is subject of future work.

Acknowledgement

The work of last author is supported in part by the TUBITAK 2221 Programme.

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