Structural Decompositions and Algorithms for the Maximum Weight Independent Set Problem

Workshop on Graph Theory and Its Applications

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Maximum Weight Independent Set problem (MWIS): given a graph $G = (V, E)$ and a weight function $w : V \to \mathbb{Q}_+$, find an independent set I in G of maximum possible weight $w(I)$, where $w(I) = \sum_{x \in I} w(x)$.

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One challenge is to identify graph classes on which MWIS can be solved efficiently.

Chordal graphs: Definition and warm-up proofs

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4. Every chordal graph admits a clique tree, i.e., a tree whose nodes correspond to the maximal cliques of the graph and satisfy the Helly property.

Perfect elimination orderings: Definition

A **perfect elimination ordering** (p.e.o. for short) of a graph G is an ordering $(v_1, \ldots, v_n)$ of its vertices such that, for every $i \in \{1, \ldots, n\}$, the set of neighbors of v_i among $\{v_1, \ldots, v_{i-1}\}$ is a clique.

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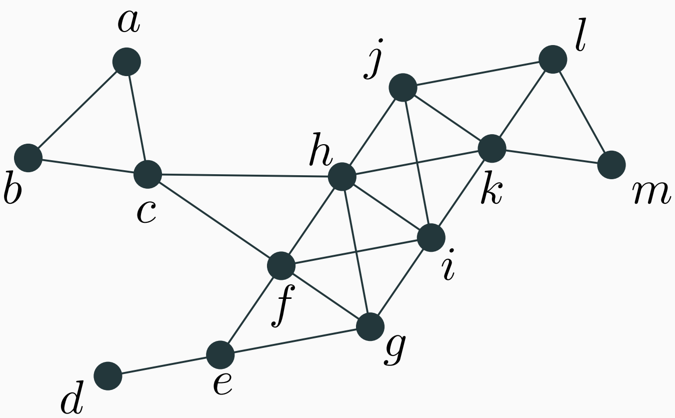

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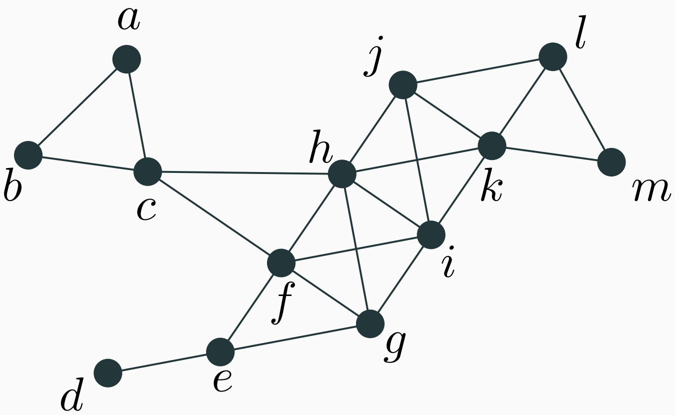

A graph is chordal if and only if it admits a p.e.o. (Rose, 1970).

Perfect elimination orderings: Solving MIS

Theorem

MIS can be solved in $\mathcal{O}(n+m)$ time on n-vertex m-edge chordal graphs using a perfect elimination ordering.

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- find a p.e.o. $(v_1, \ldots, v_n)$ of G;
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The correctness relies on the fact that at most one vertex in $N[v_i] \cap \{v_1, \ldots, v_i\}$ can be included in an independent set.

Perfect elimination orderings: Generalizing to independence degeneracy

The **independence degeneracy** (**α-degeneracy** for short) of a graph G is the minimum integer k for which there exists an ordering $(v_1, \ldots, v_n)$ of its vertices such that, for every $i \in \{1, \ldots, n\}$, the set of neighbors of v_i among $\{v_1, \ldots, v_{i-1}\}$ induces a subgraph with independence number at most k.

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There exists a $1/k$-approximation polynomial-time algorithm for MWIS on graphs with α-degeneracy at most k (Akcoglu et al., 2022).

However, we cannot expect to solve MWIS exactly in polynomial time on graphs with bounded α-degeneracy since MWIS is NP-hard to approximate within Δ^{ε}, for some $\varepsilon > 0$.

Definition

A **clique tree** of a graph G is a pair $(T, \{X_t\}_{t \in V(T)})$ such that T is a tree, each $X_t \subseteq V(G)$ is a maximal clique of G, and the following property holds:

- (Helly property) for every vertex $v \in V(G)$, the set $\{t \in V(T) : v \in X_t\}$ induces a connected subtree of T.

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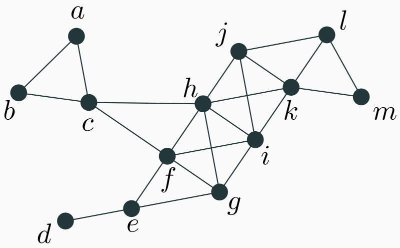
Let's show that every chordal graph admits a clique tree.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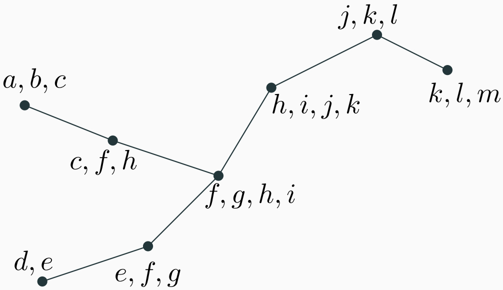
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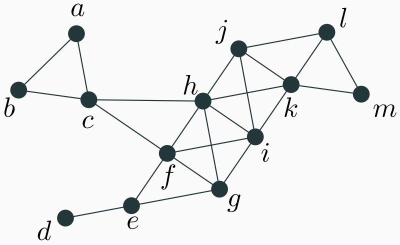

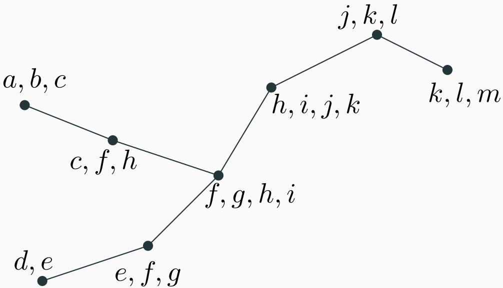

One can construct a clique tree as follows.

Invariant: Each X_t, for $t \in V(T)$, corresponds to a maximal clique of $G[\{v_1, \ldots, v_i\}]$.

- We assume that G is connected; otherwise, build a clique tree for each connected component.
- Consider a p.e.o. $(v_1, \ldots, v_n)$ of a connected chordal graph G.
- Initialize $T = (V, E)$ with $V = E = \emptyset$.
- For each i from 1 to n, do:
  - If $N(v_i) \cap \{v_1, \ldots, v_{i-1}\} = X_{t'}$ for some $t' \in V(T)$, add v_i to $X_{t'}$.
  - Otherwise, create a new node t in T and set $X_t = N[v_i]$. Add the edge tt' to E, where t' is any node of T such that $X_t \setminus \{v_i\} \subseteq X_{t'}$.
- Return $(T, \{X_t\}_{t \in V(T)})$.

Consider a chordal graph G and a clique tree $(T, \{X_t\}_{t \in V(T)})$ of G. Root the clique tree at an arbitrary node r.

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forall $t' \in N^+(t)$ **do**
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Theorem

MWIS can be solved in $\mathcal{O}(n^2)$ time on n-vertex chordal graphs using a clique tree.

Tree decompositions: Definition

Definition

A **tree decomposition** of a graph G is a pair $(T, \{X_t\}_{t \in V(T)})$ such that T is a tree, each $X_t \subseteq V(G)$ (called a bag), and the following properties hold:

- $\bigcup_{t \in V(T)} X_t = V(G)$,
- for every edge $uv \in E(G)$, there exists $t \in V(T)$ such that $\{u, v\} \subseteq X_t$, and
- for every vertex $v \in V(G)$, the set $\{t \in V(T) : v \in X_t\}$ induces a connected subtree of T.

Tree decompositions: An interesting property

Lemma

Let G be a graph and $\mathcal{T} = (T, \{X_t\}_{t \in V(T)})$ be a tree decomposition of G. Then, for every clique C of G, there exist $t \in V(T)$ such that $C \subseteq X_t$.

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Then, $N_{G^*}[u]$ is a clique in G^*, and thus there exists $t \in V(T)$ such that $N_{G^*}[u] \subseteq X_t$. $\square$

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Theorem

Given a graph G and a tree decomposition $(T, \{X_t\}_{t \in V(T)})$ of G, MWIS can be in time $O(2^w \cdot |V(T)|^2)$, where $w = \max_{t \in V(T)} |X_t|$.

Treewidth

The **width** of a tree decomposition $(T, \{X_t\}_{t \in V(T)})$ is $\max_{t \in V(T)} |X_t| - 1$, and the **treewidth** of G, denoted by $\mathrm{tw}(G)$, is the minimum width over all tree decompositions of G.

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While computing the treewidth of a graph G is NP-hard, there exists an exact FPT algorithm that either compute a tree decomposition of width k or report that $\mathrm{tw}(G) > k$, in time $2^{\mathcal{O}(k^2)} \cdot n^4$ (Korhonnen, Lokshtanov, 2023).

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The main shortcoming is that many graph classes have unbounded treewidth, including chordal graphs.

Tree-independence number

The **independence number** of a tree decomposition $(T, \{X_t\}_{t \in V(T)})$ of a graph G, denoted by $\alpha(\mathcal{T})$, is $\max_{t \in V(T)} \alpha(G[X_t])$.

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Theorem

MWIS can be solved in polynomial time on graphs with bounded tree-independence number.

From the definition of tree-independence number, it follows that chordal graphs are exactly the graphs with tree-$\alpha(G) = 1$.

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For every graph G, it holds that $\omega(G) \leq \mathrm{tw}(G) - 1$ (because every clique must be contained in some bag of any tree decomposition of G).

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Lemma

For every graph G with tree-$\alpha(G) \leq k$, it holds $\mathrm{tw}(G) < R(k+1, \omega(G)+1)$, where $R(\cdot,\cdot)$ is the Ramsey number, which is upper-bounded by a polynomial in p of degree k.

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Given a graph G, denote by $\mathcal{H}(G)$ the following graph:

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- if $\mathcal{H} = \{K_2\}$, then $\mathcal{H}(G)$ is the square of the line graph of G

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This is a common generalization of:

- the Independent $\mathcal{H}$-Packing problem (unweighted variant),
- the Maximum Weight Independent Set problem (when $\mathcal{H} = \{K_1\}$),
- the Maximum Weight Induced Matching problem (when $\mathcal{H} = \{K_2\}$),
- the Dissociation Set problem (when $\mathcal{H} = \{K_1, K_2\}$),
- the k-Separator problem (finding a minimum-weight set of vertices whose removal leaves components of size at most k).

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Sketch of proof: Let $\mathcal{T} = (T, (X_t)_{t \in V(T)})$ be a tree decomposition of G with independence number at most k. Define the tree decomposition $\mathcal{T}' = (T, (X'_t)_{t \in V(T)})$ of $\mathcal{H}(G)$ such that:

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Furthermore, an independent set in $\mathcal{H}(G)[X'_t]$, for some $t \in V(T)$, implies the existence of an independent set in $G[X_t]$ of the same size. Thus, $\alpha(G[X'_t]) \leq \alpha(G[X_t]) \leq k$. $\square$

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Theorem

The Maximum Weight Independent $\mathcal{H}$-Packing problem is solvable in polynomial time on graphs with bounded tree-independence number.

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Theorem

Let φ be a fixed $CMSO_2$ formula and let k and c be fixed integers. Then, given a graph G with tree-$\alpha(G) \leq k$, one can find in polynomial time a maximum-weight induced subgraph of G with clique number at most c that satisfies φ (if such a subgraph exists).

Operations that are monotone for tree-α

The following graph operations do not increase the tree-independence number:

- vertex deletion,

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A graph H is an induced minor of a graph G if H can be obtained from G by a sequence of vertex deletions and edge contractions.

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A graph H is an **induced minor** of a graph G if H can be obtained from G by a sequence of vertex deletions and edge contractions.

Proposition

If H is an induced minor of G, then $\text{tree-}\alpha(H) \leq \text{tree-}\alpha(G)$.

Some graph classes with unbounded tree-α

The following families of graph classes all have unbounded tree-$\boldsymbol{\alpha}$:

- bipartite graphs (tree-$\alpha(K_{n,n}) = n$),

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From the previous proposition, it follows that any graph class that contains large complete bipartite graphs, large walls, or the line graphs of large walls as induced minors has unbounded tree-α.

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- graph classes of bounded independence number (tree-$\alpha(G) \leq \alpha(G)$);
- graph classes of bounded treewidth (tree-$\alpha(G) \leq \mathrm{tw}(G) + 1$);
- classes of graphs in which all minimal separators are of bounded size (following a result by Skodinis).

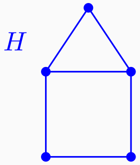
Graph containment relations

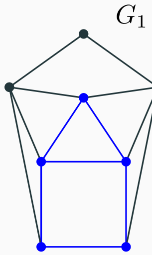
$H \subseteq_{itm} G_2$ $H \subseteq_{im} G_3$

induced subgraph

induced topological minor

induced minor





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induced
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$H \subseteq_{itm} G_2$

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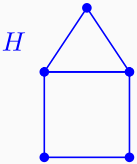

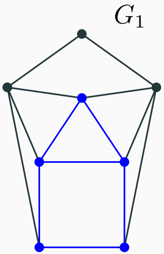

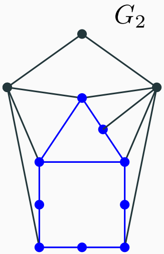

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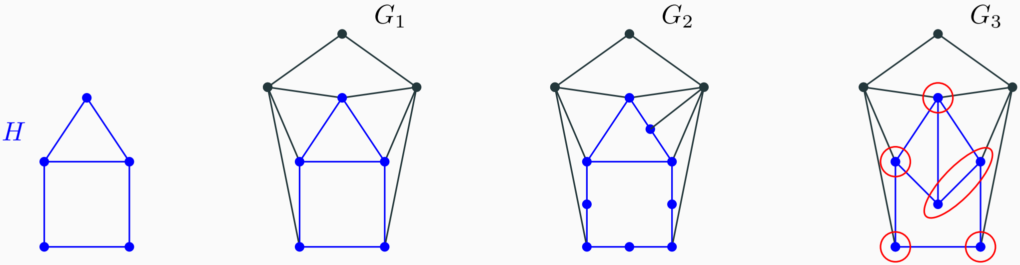

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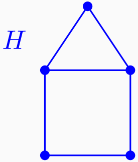
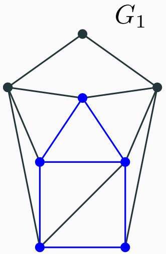
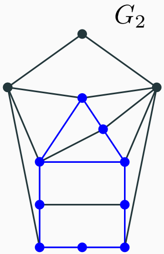




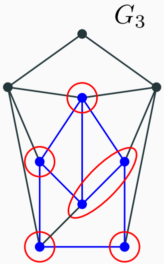
$H \subseteq_s G_1$

subgraph



$H \subseteq_{tm} G_2$

topological minor



$H \subseteq_m G_3$

minor

Forbidding a graph H

Graphs H for which the class of graphs excluding H has bounded tree-independence number (D., Milanič, Štorgel, 2024):

	Non-induced	Induced
Subgraph	$H \in \mathcal{S}$	P_3 or edgeless
Topological minor	H is subcubic and planar	C_4, K_4^-, or edgeless
Minor	H is planar	W_4, K_5^-, $K_{2,q}$ for some $q \in \mathbb{N}$

$\mathcal{S}$ is the class of graphs whose connected components are either paths or subdivisions of the claw ($K_{1,3}$).

In all the bounded cases above, we can efficiently compute a tree decomposition with bounded independence number.

Some recent works in the world of tree-α

- Treewidth versus clique number. IV. Tree-independence number of graphs excluding an induced star,
  C. Dallard, M. Krnc, O. Kwon, M. Milanič, A. Munaro, K. Štorgel, S. Wiederrecht, 2024.
- Polynomial-time approximation schemes for induced subgraph problems on fractionally tree-independence-number-fragile graphs,
  E. Galby, A. Munaro, S. Yang, 2024.
- Tree independence number II. Three-path-configurations,
  M. Chudnovsky, S. Hajebi, D. Lokshtanov, S. Spirkl, 2024.
- Tree independence number III. Thetas, prisms and stars,
  M. Chudnovsky, S. Hajebi, N. Trotignon, 2024.
- Tree independence number IV. Even-hole-free graphs,
  M. Chudnovsky, P. Gartland, S. Hajebi, D. Lokshtanov, S. Spirkl, 2024.
- Tree independence number V. Walls and claws,
  M. Chudnovsky, J. Codsi, D. Lokshtanov, M. Milanič, V. Sivashankar, 2025.
- Treewidth versus clique number. V. Further connections with tree-independence number,
  Claire Hilaire, Martin Milanič, Đorđe Vasić
- On the relation between treewidth, tree-independence number, and tree-chromatic number of graphs,
  K. Krause, M. Redzic, T. Ueckerdt, 2025.
- Excluding a Ladder as an Induced Minor in Graphs Without Induced Stars,
  M. Choi, S. Wiederrecht, 2025.
- Layered tree-independence number and clique-based separators,
  C. Dallard, M. Milanič, A. Munaro, S. Yang, 2025.
- Tree-independence number VI. Thetas and pyramids,
  M. Chudnovsky, J. Codsi, 2025.
- (Treewidth, Clique)-Boundedness and Poly-logarithmic Tree-Independence,
  M. Chudnovsky, A. E. S., D. Lokshtanov, 2025.

Beyond independence: Induced matching treewidth

For a graph G and a tree decomposition $\mathcal{T} = (T, \{X_t\}_{t\in V(T)})$ of G, let

$$\mu(\mathcal{T}) = \max_{t\in V(T)} \{M\colon M \text{ is an induced matching s.t. every edge of } M \text{ has at least one endpoint in } X_t\}.$$

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While determining tree-$\mu(G)$ is NP-hard, there exists a polynomial-time algorithm that, given a graph G and an integer k, either constructs a tree decomposition of G with induced matching number at most $\mathcal{O}(k^3)$ or reports that tree-$\mu(G) > k$ (Yolov, 2018).

Theorem (Lima et al., 2024)

Graphs with bounded tree-μ admit polynomial-time algorithms for MWIS, Max Weight Induced Forest, Max Weight Independent $\mathcal{H}$-Packing (for finite $\mathcal{H}$ of connected graphs), and more.